Theory of Higher Operads

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Abstract

This project is an exposition of Barwick's 2018 paper 'From operator categories to higher operads', which develops a theory of operads. This theory generalises May's theories of symmetric and non-symmetric operads, as well as Lurie's ∞-operads and gives us a nice way to organise $\mathbb{E}_n$-structures.

Declaration

I declare that this thesis was composed by myself and that the work contained therein is my own, except where explicitly stated otherwise in the text.

(*Lucy Spouncer*)

I dedicate this project to the incredible category theorists at Edinburgh University who have been essential in shaping my passion and understanding: Clark Barwick, Malthe Sporring and Willow Bevington.

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Introduction

This project aims to provide an exposition and critique of Clark Barwick's 2018 paper 'From operator categories to higher operads' [1]. Barwick's paper develops a theory of higher operads which builds upon much of the existing literature on operads and infinity category theory.

Let $\mathcal{O}$ be the category of ordered finite sets and $\mathcal{F}$ the category of finite sets. The structure of Barwick's theory of operads is centred around a generalisation of the categories $\mathcal{O}$ and $\mathcal{F}$ called *operator categories*. Operator categories are used to map out the intimate relationship between objects which, in various ways, 'control' algebraic structure — *operads*, *monads*, and *Kleisli categories*. In this process Barwick replaces the role of the homotopical data of topological operads, such as $\mathbb{E}_k$-operads, with the homotopical data of quasicategories. When restricting consideration only to the operator category $\Phi = \mathcal{F}$, Barwick's definition of Φ-quasioperad recovers Lurie's theory of symmetric ∞-operads [17].

We begin with classical operads (Chapter 1). Here we see topological objects such as $\mathbb{E}_k$-operads encoding all of the data for compatible algebraic structure in objects up to coherent homotopies. Alluding to the relationship between different objects that 'control' algebraic theories (Chapter 2) we obtain monoids in three different ways: via an operad, via an operator category, and via a fibration over the category Δ^{op}. Reminding the reader of the relevant infinity categorical constructions required to understand 'From operator categories to higher operads' (Chapter 3), we then turn our attention to the paper.

We follow Chapter 1 through 7 of Barwick's paper and begin by introducing the monoidal category of operator categories $\mathbf{Op}^{\wr}$ with wreath product $\wr$ (§4.1, §4.2). This product allows us to define $\mathcal{O}^{(n)} := \mathcal{O} \wr \ldots \wr \mathcal{O}$ which, via the following construction, allows us to organise $\mathbb{E}_n$-structures.

The construction is this: given a subclass of operator categories Φ with a technical property called *perfection* (§4.3) we may define a canonical monad T_Φ (§5.3); we then consider the Kleisli category of this monad $\Lambda(\Phi)$ (§5.1, §5.4); finally, the type of Φ-monoidal object we recover depends on the type of fibration we choose over $\Lambda(\Phi)$ (§5.5). This construction allows us to give objects Φ-monoidal structures, even when infinite data is required to specify the necessary algebraic operations and their coherences. Barwick's theory of operads coincides with theories of two imporant families of operads: non-symmetric operads, which are 'controlled' by the Kleisli category $\Lambda(\mathcal{O}) = \Delta^{\mathrm{op}}$, and symmetric operads 'controlled' by $\Lambda(\mathcal{F}) = \Gamma^{\mathrm{op}}$. One of the key results of the paper is that $\Lambda(\mathcal{O}^{(n)}) = \theta_n$, Joyal's disk category, such that algebras over the terminal operad of $\mathcal{O}^{(n)}$ are equivalent to $\mathbb{E}_n$-algebras.

For a perfect operator category Φ there is a subtlety in the fibrational constructions of objects with Φ-algebraic structure. At first glance Φ-operads and Φ-monoidal categories do not seem so similar, however, we will show that their definitions differ only by a class of *active* morphisms that belong to an inherent *inert-active* factorisation system on $\Lambda(\Phi)$ (§5.6).

The reader is assumed to have taken undergraduate courses in category theory and an introductory graduate course on infinity category theory or higher category theory. Knowledge of homotopy theory would be helpful but only basic concepts such as homotopy are essential.

Chapter 1

Operads

1.1 Homotopy Theory

1.1.1. Why is it important to have objects which organise algebraic theories? Short answer: sometimes there is too much essential algebraic data. For example, in the definition of the fundamental group of a space [10, p. 26], we quotient the space of loops by the homotopy relation. This allows us to consider a group structure up to a certain strict notion of equality. However, for many constructions this higher homotopical data is essential and equality is not the correct notion of equivalence. Let us look at an unsimplified version of the fundamental group, and it will become clear the resulting algebraic structure needs a comprehensive filing system.

Definition 1.1.2. For topological spaces X and Y the *compact-open* topology [10, p. 529] on the space of maps $\mathrm{Hom}(X, Y)$, is generated by subsets $\mathrm{Hom}(K, U) = f|f(K) \subseteq U$, where K is compact in X and U is open in Y.

Definition 1.1.3. Suppose we have a based topological space (X, x_0), then the *based loop space* [10, p. 395] is the topological space $\Omega X := \mathrm{Hom}((S^1, *), (X, x_0))$ whose points are loops in X containing x_0 endowed with the compact-open topology .

Construction 1.1.4. This loop space has a natural product, namely concatenation of loops [5, Def. 14.11]. Elements of ΩX may be identified with continuous maps l in $\mathrm{Hom}_*([0, 1], (X))$ such that $l(0) = l(1) = x_0$. Given two such loops $l_1, l_2 : [0, 1] \to X$, define their concatenation as follows:

$$l_1 \bullet l_2(t) := \begin{cases} l_1(2t), & 0 \leq t \leq 0.5, \\ l_2(2t - 1), & 0.5 \leq t \leq 1. \end{cases}$$

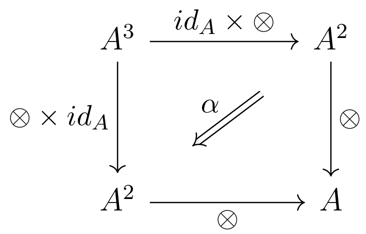
Due to the parameterisation of our concatenation we do not get equality of $(l_1 \bullet l_2) \bullet l_3$ with $l_1 \bullet (l_2 \bullet l_3)$. Nor do we get equality for the unitality condition considering concatenation of loop l_i with the constant loop 1_{x_0} at x_0. Instead, we have homotopies

$$\alpha_{l_1,l_2,l_3} : (l_1 \bullet l_2) \bullet l_3 \xrightarrow{\simeq} (l_1 \bullet l_2) \bullet l_3$$

and unitor

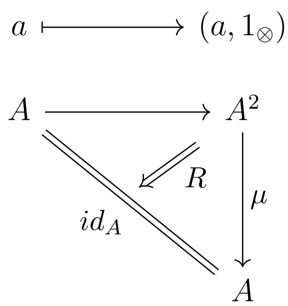
$$L_{l_i} : 1_{x_0} \bullet l_i \to l_i$$
$$R_{l_i} : l_i \bullet 1_{x_0} \to l_i.$$

Definition 1.1.5. Given an operation $\bigotimes : A \times A \to A$ we may not be able to ask that it is *strictly* associative. If A belongs to a 2-category it makes more sense to ask instead that the operation is associative *up to 2-isomorphism.* If A is a topological space, we might have associativity *up to homotopy.* In this case we call the 2-isomorphism or homotopy the *associator*, labelled α in following commutative diagram [20, p. 126].



(1.1)

For all $a, b, c \in A$, the associator in (1.1) gives a map $a \otimes (b \otimes c) \mapsto (a \otimes b) \otimes c$. Similarly, the 2-isomorphisms or homotopies that give the unitality of the operation are called the left and right *unitors.* The following diagram gives the right unitality of the product up to its right unitor R.



(1.2)

Remark 1.1.6. Continuing in this vein, we might consider two homotopies between a pair of loops and ask if those homotopies are homotopic to one another. We find that conditions such as associativity and unitality of a product such as concatenation are no longer properties, but additional structure which must be specified. The following section begins to solve the problem of specifying the infinite compatibility data of algebraic structures up to homotopy.

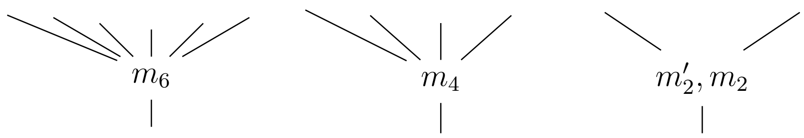
1.2 Classical Operads

Definition 1.2.1. An *operad* [15, Def. 2.2.1] in $(Set, \times, 1)$ is:

- a sequence of sets $(P(n))_{n\in\mathbb{N}}$;
- functions $P(n) \times P(k_1) \times ...P(k_n) \to P(k_1 + ... + k_n)$ for each $n \in \mathbb{N}$ given by $(\theta, \theta_1, ..., \theta_n) \mapsto \theta \circ (\theta_1, ..., \theta_n)$;
- a distinguished element $id \in P(1)$.

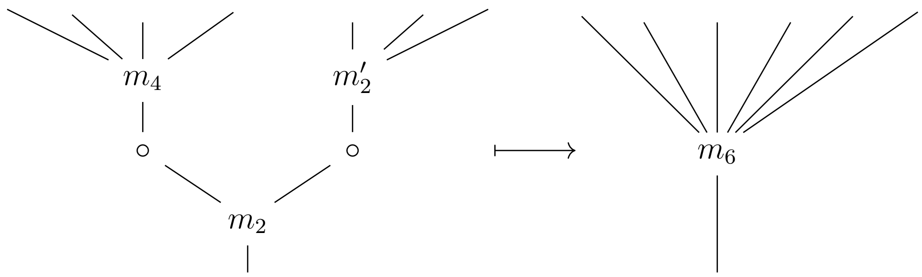
Here elements of $P(n)$ correspond to n-ary operations and id is the identity unary operation. A morphism of operads $F : Q \to P$ is a family of functions $F_n : Q(n) \to P(n)$ such that identity and composition of operads are preserved [15, p. 43].

Observation 1.2.2. We may visualise the n-ary operations as trees where the leaves are inputs and the stem an output. For example, given $m_2, m_2' \in P(2)$, $m_4 \in P(4)$, and $m_6 \in P(6)$ we have the trees in the diagram below.



(1.3)

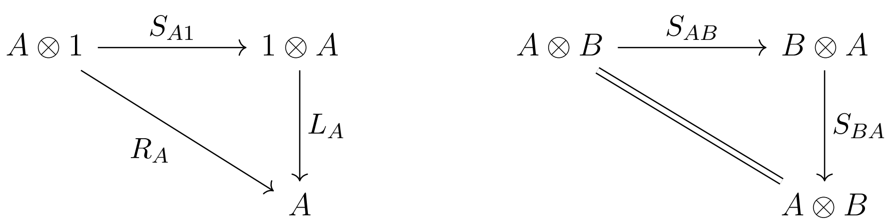
Suppose that composition of such operations is given by the map $\circ : P(2) \times P(4) \times P(2) \to P(6)$ which takes $(m_2, m^4, m_2') \mapsto m_6$ as below.



Definition 1.2.3. A (symmetric) monoidal category $\mathcal{C}^{\otimes}$ is determined by the following data and properties [4, Def. 6.1.1-2]:

- a category $\mathcal{C}$;
- a functor $\bigotimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$;
- a functor $1_\otimes : \mathbb{1} \to \mathcal{C}$ picking out the identity object;
- four natural isomorphisms, the associator α, left and right unitors R and L as introduced in Definition 1.1.5 and the symmetriser S,

such that natural isomorphism S with components $S_{AB} : A \otimes B \to B \otimes A$ satisfies the following commutative diagrams.



Remark 1.2.4. We may, in fact, define operads over any symmetric monoidal category [15, p. 43] and it is often useful to work in categories other than **Set** .

Example 1.2.5. Let $\mathbf{Vect}^\otimes$ be the category of vector spaces over field $\Bbbk$ with the usual tensor product $\bigotimes : \mathbf{Vect}^2 \to \mathbf{Vect}$. Specifying the associator and unitors below gives us that $\mathbf{Vect}^\otimes$ is a symmetric monoidal category. For vector spaces Y, W, Z let the component $\alpha_{Y,W,Z}$ of the associator α be the isomorphism below:

$$\mathrm{Hom}(((Y \otimes W) \otimes Z), X) \cong \mathrm{Hom}((Y \otimes (W \otimes Z)), X)$$

via repeated use of the tensor $\dashv$ hom adjunction [13, Ex. 5.7]. In the same way, the components of the right unitor $R_{W,Z}$ are given by the following isomorphisms:

$$\mathrm{Hom}(\Bbbk \otimes W, Z) \cong \mathrm{Hom}(\Bbbk, \mathrm{Hom}(W, Z)) \cong \mathrm{Hom}(W, Z),$$

and similarly for the left unitor L. The object $\mathbf{Vect}^\otimes$ will be a core example throughout this project and each instance of it will also be over the (arbitrary) ground field $\Bbbk$.

Definition 1.2.6. Let $\mathcal{C}$ be an operad in the symmetric monoidal category $(\mathcal{V}, \otimes, 1)$. An *algebra over operad* (adapted from [15, Def. 2.1.12]) $\mathcal{C}$, consists of the following:

- an object $A \in \mathcal{V}$;
- a family of morphisms in $\mathcal{V}$, $(P(n) \otimes A^{\otimes n} \to A)_{n \in \mathbb{N}}$,

such that the morphisms are compatible with the composition functions $\circ$ in the operad P, as in the commutative diagram below.

$$\begin{array}{ccc} P(n) \times P(k_1) \times \cdots \times P(k_n) \times A^{k_1+...+k_n} & \longrightarrow & A \\ \downarrow {\scriptstyle \circ \times id} & \nearrow & \\ P(k_1 + ... + k_n) \times A^{k_1+...+k_n} & & \end{array}$$

Remark 1.2.7. Operads give us an abstract way to encode all allowed operations in types of algebraic theories. Taking algebras over operads give us instances of objects which express some of this algebraic data. The morphisms determine the structure of the product and the objects in Definition 1.2.6 determine the possible inputs and outputs of operations.

1.3 Operads as Multicategories

1.3.1. The reader may remember when she first learnt about categories and came across the definition of a monoid as a one object category. They quickly would have realised that although the mathematics of one object categories is very rich, that of multiobject categories is infinitely richer. We encounter the same idea with operads as follows.

Definition 1.3.2. A *multicategory* $\mathcal{C}$ is defined by the following data [15, Def. 2.1.1]:

- multicategory $\mathcal{C}$ has a collection of objects $Ob(\mathcal{C})$;
- for each object b and family of objects $(a_i)_{i \in \langle n \rangle}$ in $\mathcal{C}$ there is a collection of morphisms $\mathcal{C}(a_1, ..., a_n; b)$;

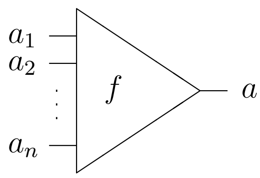


Figure 1.1: $f \in \mathcal{C}(a_1, ..., a_n; b)$

- for all $n \in \mathbb{N}$ and $a, a_i \in \mathcal{C}$ an identity element $1 \in \mathcal{C}(a_1, ..., a_n; a)$;

- for each $n, k_1, ..., k_n$ and objects $a, a_i, a_i^j \in \boldsymbol{\mathcal{C}}$ a function called *composition* $\boldsymbol{\mathcal{C}}(a_1, ..., a_n; A) \times \boldsymbol{\mathcal{C}}(a_1^1, ..., a_1^{k_1}; a_1) \times ... \times \boldsymbol{\mathcal{C}}(a_n^1, ..., a_n^{k_n}; a_n)$ $\to \boldsymbol{\mathcal{C}}(a_1^1, ..., a_1^{k_1}, ..., a_n^1, ..., a_n^{k_n}; a)$ written $(\theta, \theta_1, ..., \theta_n) \to \theta \circ (\theta_1, ..., \theta_n)$;

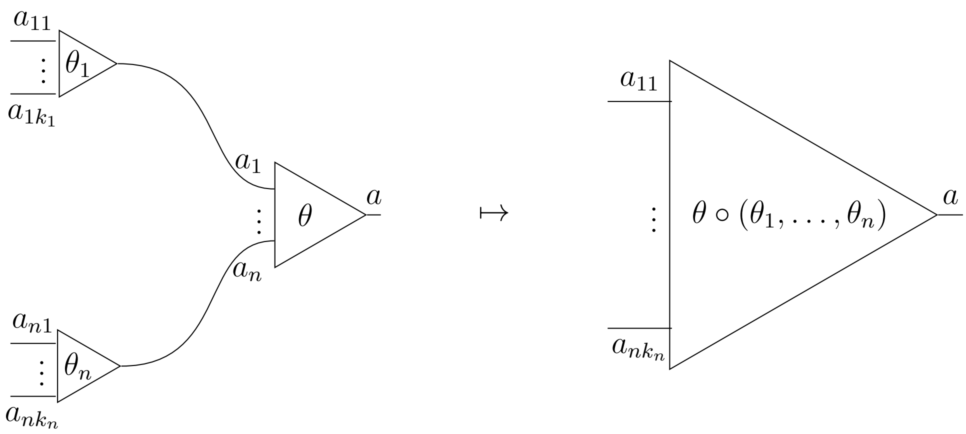


Figure 1.2: Multicategory composition [15, Fig.2-B]

where the composition satisfies obvious associativity and identity laws.

Example 1.3.3. The category of vector spaces **Vect** with $\mathbf{Vect}((W_i)_{i\in\langle n\rangle}; V) = \{m : W_1, ..., W_n \to V |\ m \text{ is multilinear}\}$ is a multicategory.

Notation 1.3.4. For $\boldsymbol{\mathcal{F}}$ the category of finite sets and $\boldsymbol{\mathcal{F}}_+$ the category of pointed finite sets, we will let $\langle n\rangle \in \boldsymbol{\mathcal{F}}$ denote the set of n elements and $\langle n\rangle_+ \in \boldsymbol{\mathcal{F}}_+$ denote the set $\{0, ..., n\}$ with distinguished point $0 \in \langle n\rangle_+$.

Definition 1.3.5. Taking the symmetric monoidal category **Vect** and following [17, Cons. 2.0.0.1] we conclude that $\mathbf{Vect}^{\otimes}$ has tuples of vector spaces as objects. We also have morphisms $\alpha_f : (C_1, \cdots, C_n) \to (D_1, \cdots, D_m)$ for vector spaces $C_i, D_i \in \mathbf{Vect}$ given by the following tuple:

$$(\alpha : S \subseteq \langle n\rangle \to \langle m\rangle, (f_j : \bigotimes_{\alpha(i)=j} C_i \to D_j)_{j\in\langle m\rangle}) \tag{1.4}$$

Composition is detailed as follows: say we have two morphisms

$$(\alpha : S \subseteq \langle n\rangle \to \langle m\rangle, (f_i)_{i\in\langle m\rangle}),\ (\beta : T \subseteq \langle m\rangle \to \langle k\rangle, (g_j)_{j\in\langle m\rangle})$$

then let the composite be

$$(\beta \circ \alpha : U \subseteq \langle n\rangle \to \langle k\rangle, ((g \circ f)_i)_{i\in\langle k\rangle}$$

where $U = \alpha^{-1}(T)$ and we have

$$(f \circ g)_i : \bigotimes_{\beta(j)=t} \bigotimes_{\alpha(t)=i} C_j \xrightarrow{(g_t \circ f_i)_{t\in\langle m\rangle}} B_i.$$

Example 1.3.6. The definition of $\mathbf{Vect}^{\otimes}$ lends itself naturally to that of a multicategory comprising the following data:

- one object $\mathbf{Vect}$;
- for all $n \in \mathbb{N}$

  $$\boldsymbol{\mathcal{V}}(\mathbf{Vect}^n; \mathbf{Vect}) := \{(\alpha, (f_i)_{i\in\langle n\rangle}) : \mathbf{Vect}^n \to \mathbf{Vect} | \alpha : S \subseteq \langle n\rangle \to \langle 1\rangle\}$$

  the set of morphisms of the form in (1.4);
- composition is the same as above where precomposing with a morphism $(\alpha : S \subseteq \langle m\rangle \to \langle n\rangle, (f_i)_{i\in\langle n\rangle})$ is interpreted as precomposing with $\langle n\rangle$ morphisms of our mulitcategory.

Observation 1.3.7. Classical operads may alternatively be defined as one object multicategories [15, Def. 2.2.1]. Call the unique object $*$ then in the operad P the sets $P(n)$ are the multihom sets of the morphisms from n copies of $*$ to itself. Therefore, the definition of multicategory generalises both classical operads and ordinary categories; fantastic, now we have multimorphisms and multiple objects. From now on we will refer to one-object multicategories as classical operads and multicategories as operads.

1.4 Operads $\mathbb{E}_k$

1.4.1. Let us introduce a particular representation of important class of classical operads in $\mathbf{Top}^{\amalg} := (Top, \amalg, 1 := \{\})$ the symmetric monoidal category of topological spaces with disjoint union as the product. In this context, operations correspond to continuous maps and have their own built in notion of homotopy.

Definition 1.4.2. The *operad* $\mathbb{E}_k$ [17, Def. 5.1.0.2] for $k \in \mathbb{N}$ consists of the spaces $\mathbb{E}_k(n)$ of disjoint, rectilinear embeddings of n many k-dimensional cubes into a k-dimensional cube for each $n \in \mathbb{N}$. These embeddings parameterise the n-ary operations of $\mathbb{E}_k$.

Remark 1.4.3. The disjointness condition in Definition 1.1.2 is what gives our spaces of operations non-trivial higher homotopies.

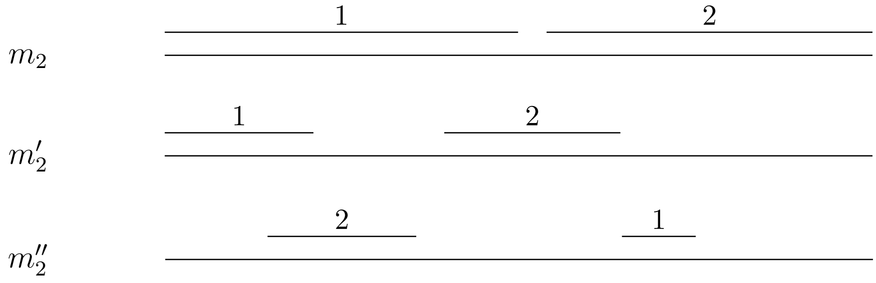


Figure 1.3: Three embeddings m_2, m_2' and m_2'' of unit intervals 1 and 2 into the unit interval.

Example 1.4.4. The space $\mathbb{E}_1(n)$ is the space of disjoint embeddings of n intervals into an interval. Consider three binary operations m_2, m_2', and m_2'', points in $\mathbb{E}_1(2)$.

The topology of $\mathbb{E}_1(2)$ tells us that m_2 and m_2' are homotopic, since we may continuously deform the embedding m_2 to the embedding m_2'. However, neither are homotopic to m_2'' as we may not continuously pass the embedded intervals through each other. And so, these n-ary operations define a single non-commutative, associative and unital product, but only up to homotopy [17, p. 757].

Observation 1.4.5. We may consider these embeddings as parameterising concatenations of loops in our based loop space (X, x_0) in the following way [17, p. 870]. Given loops $l_i : [0,1] \to X$ for i in set S, define their concatenation with respect to the embeddings J_i as components of $J : \amalg_{i \in S} I \to I$ to be

$$\bullet_{J_S} l_S = \begin{cases} J_i(l_1(\frac{t - J_i(0)}{|J_i(I)|})), & t \in J_i(I) \\ x_0, & \textit{otherwise.} \end{cases}$$

In the diagram 1.4.4, m_2 corresponds to traversing l_1 in first half the interval and l_2 in the second.

Example 1.4.6. A based loop space ΩX is an algebra over operad $\mathbb{E}_1$ [17, p. 870], where we define $(P(|S|) \otimes A^{\otimes S} \to A)_{S \in \mathbf{Set}}$ by $J \otimes (l_i)^{\otimes S} \mapsto \bullet_{J_S} l_S$.

Observation 1.4.7. To clarify the following explanation we will call homotopies between operations 1-homotopies, homotopies between 1-homotopies will be 2-homotopies, and so on. Operations that determine the product in an $\mathbb{E}_2$-algebra are parameterised by disjoint embeddings of squares into a square. Unlike $\mathbb{E}_1$ this product is commutative up to 1-homotopy, however, as seen in Figure 1.4 there are multiple 1-homotopies between embeddings which themselves are not 2-homotopic. Let us say that $a \cdot b$ is the binary operation parameterised by the embedding in

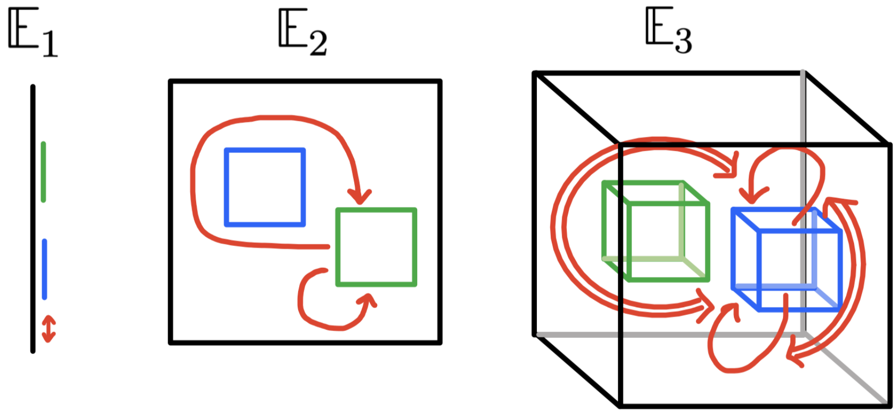


Figure 1.4: Representations of operads $\mathbb{E}_1$, $\mathbb{E}_2$, and $\mathbb{E}_3$.

the figure above. Consider two pairs of 1-homotopies

$$\circlearrowright_1, \circlearrowleft_1 \colon a \cdot b \to b \cdot a, \text{ and } \circlearrowright_2, \circlearrowleft_2 \colon b \cdot a \to a \cdot b,$$

which in each case interchange the two squares moving them around each other clockwise and anticlockwise respectively. Then $\circlearrowleft_1 \circ \circlearrowright_2 \colon a \cdot b \to a \cdot b$ is 2-homotopic to the constant 1-homotopy, but $\circlearrowleft_1 \circ \circlearrowleft_2 \colon a \cdot b \to a \cdot b$ is Not. For $\mathbb{E}_3$ all binary operations are 1-homotopic, and all 1-homotopies between them are 2-homotopic, however, their 3-homotopies may not be 4-homotopic!

Example 1.4.8. If we take a loop space ΩX and we consider the space of its loops $Y := \Omega^2 X$, then Y, with concatenation of loops as the product, has an $\mathbb{E}_2$-structure [17, p. 870].

Chapter 2

Three Ways to Define a Monoid

2.0.1. In this chapter we will consider non-symmetric structures, namely (non-commutative) monoids defined in sets and categories.

2.1 Algebras Over the Terminal Operad $\mathbb{1}_O$

Construction 2.1.1. Considering morphisms between operads in **Set** then we see that the terminal (non-symmetric) operad $\mathbb{1}_O$ is the one object operad where each $\mathbb{1}_O(n)$ is a singleton, each composition function $\mathbb{1}_O(n) \times \mathbb{1}_O(k_1) \times ... \times \mathbb{1}_O(k_n) \to \mathbb{1}_O(k_1 + ... + k_n)$ is the unique map.

Definition 2.1.2. A *monoid* in **Set** is an $\mathbb{1}_O$-algebra in **Set** [15, Ex. 2.1.11]. Let $(s_1...s_n)$ denote 'the' n-fold product, where $n = 0$ gives the identity element of M. Then a monoid comprises a set M such that for $s_i^j \in M$ we have

$$(s_1^1, ..., s_1^{k_1})(s_2^1, ..., s_2^{k_2})...(s_n^1, ..., s_n^{k_n}) = (s_1^1...s_n^{k_n})$$

.

Remark 2.1.3. Since the n-ary operation $(s_1, ..., s_n) \mapsto s_1...s_n$ is unique for each $n \in \mathbb{N}$ we arrive at algebraic operations which exhibit strict non-commutativity, unitality and associativity.

Example 2.1.4. A monoid in the category of monoids **Mon** is a commutative monoid. Take $M \in$ **Mon** with monoid homomorphisms $\mathbb{1}_O(n) \times M^n \to M$. Take $\bullet$ to be the monoidal product in M and denote the monoidal product determined by operad $\mathbb{1}_O$ by $\mu : M^2 \to M$ where $(a, b) \mapsto a \otimes b$. Monoid morphisms preserve identity, so the nullary operation $* \in \mathbb{1}_O(0)$ picks out $1_\bullet \in M$ identifying the units of the monoidal products $1 := 1_{\otimes} = 1_{\bullet}$. The monoidal product is preserved by monoid homomorphisms, and so we get the exchange law

$$(a \bullet b) \otimes (c \bullet d) = (a \otimes c) \bullet (b \otimes d).$$

Substituting in the monoidal unit into the exchange law gives us that $\otimes = \bullet$ is commutative. Thus M is a commutative monoid. This is known as the Eckman-Hilton argument [6, p. 244].

Remark 2.1.5. We can think of this terminal operad as having an operation for each linearly ordered set I, there is an important operad which generalises this idea.

Definition 2.1.6. The associative operad Assoc has one object, and a morphism in Assoc(n) for each linear ordering of the n-element set I.

Lemma 2.1.7. *For finite set F and unit interval $I := [0, 1] \in \mathbb{R}$, the space of rectilinear disjoint embeddings Rect$(\amalg_{i \in F} I_i, I)$ is homotopy equivalent to the discrete set of linear orderings of F.*

Proof. See [17, Ex. 5.1.0.7]. □

Remark 2.1.8. Further comment in [17, Ex. 5.1.0.7] says that this homotopy equivalence means that as topological operads there is some sort of weak equivalence between Assoc and E_1.

2.1.9. To conclude this section we give a comparison of the following non-symmetric operads:

- the terminal operad gives us monoids in the strict sense;
- the operad Assoc 'controls' the theory of strictly associative algebras [17, Rem. 4.1.1.2]; and
- the operad $\mathbb{E}_1$ gives us objects with non-symmetric monoidal structure up to homotopy.

2.2 Pullback over Δ^{op}

Notation 2.2.1. Let $[n]$ denote the linearly ordered set $\{0 < 1 < ... < n\}$, objects of the simplex category Δ.

Construction 2.2.2. Let $P : \mathbf{Vect}^{\otimes} \to \Delta^{\text{op}}$ be a functor which does the following [17, p. 166-167]:

- sends objects $\mathbf{Vect}^n$ to $[n]$;

- sends morphism $\phi = (\alpha, (f_j)_{i\in\langle m\rangle})$ as in 1.3.5 to morphism $[[m] \xrightarrow{\bar{\phi}} [n]]$ for

$$\bar{\phi}(i) = \begin{cases} \sum_{k=1}^{i} |\alpha^{-1}(i)|, i \neq 0 \\ 0, i = 0. \end{cases} .$$

Remark 2.2.3. Then symmetric monoidal product of $\mathbf{Vect}^{\otimes}$ may be recovered via the pullback construction below [17, Rem. 2.0.0.6].

$$\begin{array}{ccc} [\bigotimes : Vect^2 \to Vect] & \longrightarrow & \mathbf{Vect}^{\otimes} \\ \downarrow & & \downarrow p \\ [[2] \to [1]] & \longrightarrow & \Delta^{\mathrm{op}} \end{array}$$

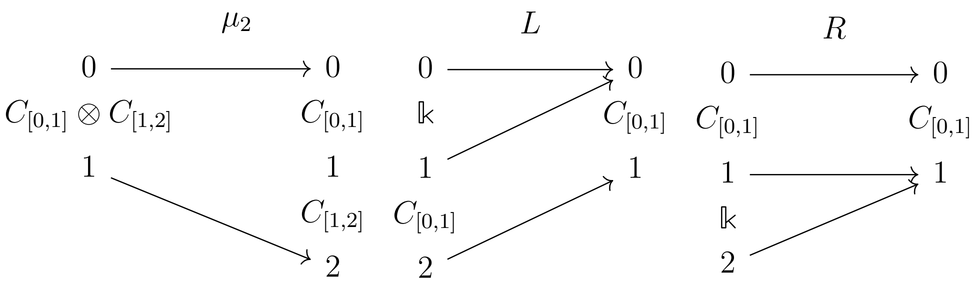
In general, we can construct a functor between fibres $\mathbf{Vect}^n \to \mathbf{Vect}^m$ for every morphism $[m] \to [n] \in \Delta^{\mathrm{op}}$.

Construction 2.2.4. Take $\phi : [n] \to [m] \in \Delta^{\mathrm{op}}$, let α be the induced map on edges $\{i-1 < i\} \overset{i}{\hookrightarrow} [n]$. Then for each lift $(C_{[0,1]}, ..., C_{[m-1,m]})$ of $[m]$ in p we have a special lift of ϕ [17, (M1) p.166]

$$f := (\alpha : \langle m\rangle \to \langle n\rangle, (f_j : \bigotimes_{\alpha(i)=j} C_i \xrightarrow{id} \bigotimes_{\alpha(i)=j} C_i)_{j\in\langle m\rangle}). \tag{2.1}$$

It is special in that it determines the image of the functor $\mathbf{Vect}^n \to \mathbf{Vect}^m$ obtained by pulling back over the inclusion $\phi \hookrightarrow \Delta^{\mathrm{op}}$. In this way we can recover the left and right unitors and associators of the monoidal product $\bigotimes$ identical to those in Example 1.2.5.

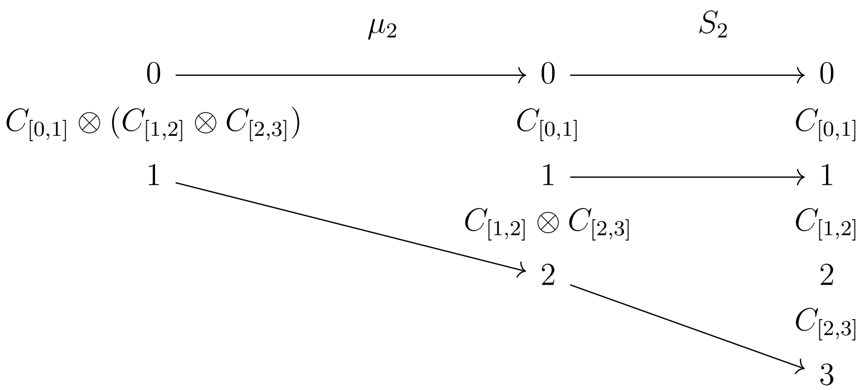
Construction 2.2.5. Let μ_n denote the morphism $[1] \to [n] \in \Delta$ where $0 \mapsto 0$ and $1 \mapsto n$. The pullback of μ_n is $\mu_n^* : \mathbf{Vect}^n \to \mathbf{Vect}$ which sends $(C_{[0,1]}, ..., C_{[n-1,n]} \mapsto (C_{[0,1]} \otimes ... \otimes C_{[n-1,n]})$. We obtain the properties of the left and right unit by considering the unique isomorphisms between pullbacks over the following composites: $\mu_2 \circ L$; $\mu_2 \circ R$; and the identity $[1] \xrightarrow{id} [1]$ for L and R in the diagram below.



(2.2)

In diagrams 2.2 and (2.3) we let the arrows indicate the map in Δ and the induced map on edges in the opposite direction corresponds to the special lift which defines the associated functor.

Construction 2.2.6. Next we want to consider associativity. Let the pullback over $S_1 : [3] \to [2] \in \Delta$ be the functor $\mu_2^* \times id$, and the pullback over $S_2 : [3] \to [2] \in \Delta$ be $id \times \mu_2^*$. Associativity is recovered by taking the pullback of composites $S_2 \circ \mu_2$ (see diagram (2.3)), $S_1 \circ \mu_2$, and μ_3 and considering the isomorphisms between them as with unitality above.



(2.3)

Lemma 2.2.7. *For vector space $C_{[0,1]} \in$ **Vect**$^{\otimes}$ as in Construction 2.2.2 we have the following isomorphisms:*

(1) $C_{[0,1]} \otimes \Bbbk \cong C_{[0,1]} \cong \Bbbk \otimes C_{[0,1]}$*; and*

(2) $(C_{[0,1]} \otimes C_{[1,2]} \otimes C_{[2,3]}) \cong C_{[0,1]} \otimes (C_{[1,2]} \otimes C_{[2,3]}) \cong (C_{[0,1]} \otimes C_{[1,2]}) \otimes C_{[2,3]}$.

Proof. By the Yoneda Lemma, the isomorphic functors in Construction 2.2.5 and 2.2.6 above have isomorphic representing objects. ☐

2.3 Category $\mathcal{O}$ of Ordered Finite Sets

Construction 2.3.1. There is another object that seems to control the theory of non-commutative monoids in a different way—the category $\mathcal{O}$ of ordered finite sets (adapted from [1, p. 1894]). For a set $M \in$ **Set** let each ordered finite set $I \in \mathcal{O}$ correspond to an $|I|$-ary operation $M^{|I|} \xrightarrow{\phi_I} M$. Here $M^{|I|}$ is the $|I|$-fold cartesian product of M with itself and ϕ_I is just a function. The nullary operation $M^0 \to M$ picks out the unit in M. Then the monoidal structure on M is determined by the following commutative square, for each $\eta : J \to I \in \mathcal{O}$, with fibres $J_i := \{i\} \times_I J$ for $i \in I$.

$$\begin{array}{ccc} M^{\sum_{i\in I}|J_i|} & \xrightarrow{=} & M^{|J|} \\ {\scriptstyle (\phi_{J_1} \times \ldots \times \phi_{J_{|I|}})}\downarrow & & \downarrow{\scriptstyle \phi_J} \\ M^{|I|} & \xrightarrow{\phi_I} & M \end{array} \tag{2.4}$$

Observation 2.3.2. Here we see that the data of the left and right unit of the monoidal product are given by the two inclusions $\{1\} \hookrightarrow \{1, 2\}$, and associativity is determined by surjections $\{1, 2, 3\} \to \{1, 2\}$. If we consider instead the category of finite sets $\mathcal{F}$, we also have that involutions $\{1, 2\} \to \{1, 2\} \in \mathcal{F}$ give us commutativity of the monoidal product. [1, p. 1894]

Remark 2.3.3. This is a commuting square of morphisms in an ordinary category and so we recover an object with strict associativity and unitality.

2.3.4. It is worth noting that the commuting squares in this section and the previous feel very similar and seem to suggest a nice relationship between the categories Δ^{op} and $\mathcal{O}$. Before we can explicate this relationship we must remind the reader of important infinity categorical constructions needed to understand Barwick's paper on operator categories — the main content of this project's exposition.

Chapter 3

Infinity Categories and their Constructions

3.1 Simplicial Sets and Infinity Categories

3.1.1. Barwick's paper [1] uses a particular model for infinity categories which utilises the special properties of the category Δ, the category of simplices [20, p. 175]. We begin this chapter by giving the reader the tools to consider objects with simplicial properties.

Definition 3.1.2. [20, p. 178] A *simplicial object* in category $\mathcal{C}$ is a functor $X : \Delta^{\text{op}} \to \mathcal{C}$. Where $\mathcal{C} = \mathbf{Set}$ we call X a *simplicial set*.

Remark 3.1.3. We can think of simplices geometrically as in the following figure; $[n]$ may be viewed as the space $\{\sum_{i\in[n]} x_i \leq 1 | 0 \leq x_i \in \mathbb{R}^{n+1}\} \in \mathbb{R}^{n+1}$ [20, p. 178].

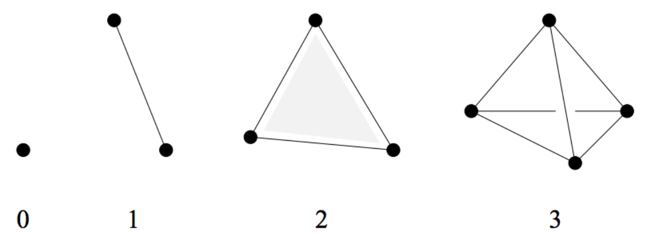


Figure 3.1: Simplices in dimensions 0 to 3. [23, Fig.1]

Specifically in infinity category theory, simplicial objects help us encode and express homotopical information combinatorially.

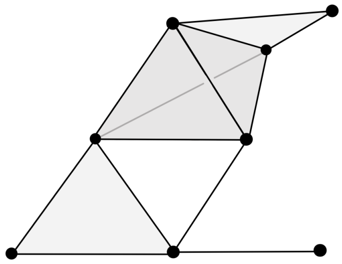


Figure 3.2: Interpretation of a simplicial object with one 3-simplex, two 2-simplices, 13 1-simplices and eight 0-simplices.

3.1.4. The image of simplicial set X on $[n]$ is the set of n-simplices of our simplicial object. For intuition, at least at first, it suffices to visualise them as built from 'gluing' simplicies together along subsimplices as in Figure 3.2.

In addition, we may transition between thinking about simplicial sets and categories by interpreting commutative diagrams as simplices. An n-tuple of composable morphisms can be visualised as an n-simplex, where we fill in all possible subcomposite maps (see diagram 3.3).

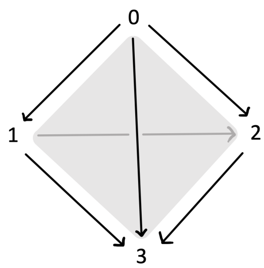


Figure 3.3: Composable 3-tuple of morphisms as a [3]-simplEx.

Construction 3.1.5. We obtain a face of an n-simplex by removing a point i and its attaching subsimplices. We call this the i^{th} face. The $\wedge^n_k$ horn of an n-simplex is the union of all faces except for the k^{th} face [19, Cons. 1.2.4.1]. We will call $\wedge^n_k$ an *inner horn* if $0 < k < n$, and an *outer horn* if $k = 0$ (left horn) or $k = n$ (right horn).

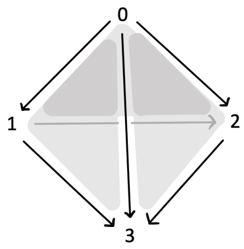


Figure 3.4: A 3-simplex with shaded left outer horn $\wedge_0^3$.

Definition 3.1.6. Let X be a simplicial set.

$$\begin{array}{ccc} \wedge_k^n & \xrightarrow{g} & X \\ \big\downarrow & \nearrow_{f} & \\ \Delta^n & & \end{array}$$

If the diagram above commutes then f is called a *horn filler*. If for every map $\wedge_k^n \to X$ there exists an extension $\Delta^n \to X$ then X satisfies the $\wedge_k^n$ *horn filler condition* (often called the *Kan extension condition* [18, p. x, 8]).

Definition 3.1.7. If simplicial set X satisfies the $\wedge_k^n$ horn filler condition for all $n \geq 2$ and $n \geq k \geq 0$ then X is an ∞-*groupoid*. This definition coincides with the *Kan complex* model for ∞-groupoids [18, Def. 1.1.2.1].

Definition 3.1.8. For simplicial set X, if the $\wedge_k^n$ horn filler condition is satisfied for $0 < k < n$, $n \geq 2$ then X is an $(\infty, 1)$-*category* [18, Def. 1.1.2.4]. Here we have given the *quasicategories* model of $(\infty, 1)$-categories.

Definition 3.1.9. A functor between quasicategories, an *$(\infty$, 1)-functor*, is simply a morphism of the underlying simplicial sets. From this point we will use the term ∞-*category* to mean quasicategory and, where the ∞-categorical context is clear, we shall just refer to an $(\infty, 1)$-functor as a functor.

In the following definition, the functor we introduce gives us an important bridge from ordinary categories to quasicategories.

Definition 3.1.10. The *nerve functor* N on categories [18, p. 9] is given by

$$\begin{aligned} N(-) : \mathbf{Cat} &\to \mathbf{sSet} \\ \mathcal{C} &\mapsto [\Delta^{\text{op}} \xrightarrow{N(\mathcal{C})} \mathbf{Set}], \end{aligned}$$

where the simplicial set is specified by

$$\begin{aligned} N(\mathcal{C}) : \Delta^{\mathrm{op}} &\to \mathbf{Set} \\ [n] &\mapsto \mathbf{Fun}([n], \mathcal{C}). \end{aligned}$$

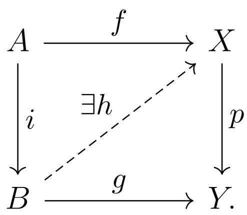
Definition 3.1.11. The *nerve of a category* $N\mathcal{C}$ [18, p. 9] is the simplicial set $N\mathcal{C}$ in the image of $\mathcal{C}$ under the nerve functor N.

Observation 3.1.12. Suppose $\mathcal{C}$ is an ∞-category then the Λ^n_k-horn filler condition is the assertion that $\mathrm{Hom}(\Delta^n, X) \to \mathrm{Hom}(\Lambda^n_k, X)$, precomposition with the inclusion, is a surjective map. In other words, for every composable tuple of morphisms, there exists a composite morphism that completes the commutative diagram in the image of Λ^n_k. If we ask for this map to be a bijection then we demand that this composite is unique, that $\mathcal{C}$ is equivalent to the nerve of an ordinary category [18, Prop. 1.1.2.2].

3.2 Fibrations

3.2.1. In order to get a handle on the infinite amounts of data in infinity categories we use an analogue of fibre bundles as in differential geometry [14, p. 268]. We take a base simplicial set with a map into it whose fibres are also simplicial sets. This enables us to lift desirable properties of the base to the collection of fibres.

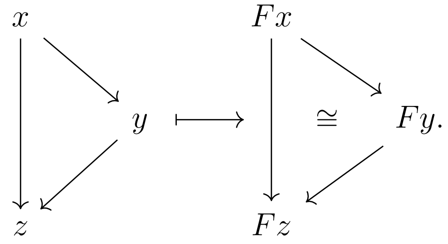
Definition 3.2.2. The morphism p has the *right lifting property* [12, Def. 1.3] with respect to morphism i if, whenever the outer square of the diagram below commutes, there exists a diagonal morphism h which makes the whole diagram commute.



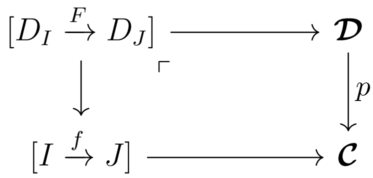
Definition 3.2.3. Consider a morphism of simplicial sets $p : X \to Y$ then p is an *inner fibration* [18, Def. 2.0.0.3] if it satisfies the right lifting property with respect to all inner horn inclusions.

Example 3.2.4. Every functor between ordinary categories $\mathcal{C} \to \mathcal{D}$ induces an inner fibration of nerves $N(\mathcal{C}) \to N(\mathcal{D})$ (as implied by [18, Prop. 1.1.2.2]).

3.2.5. Consider the case were we want to specify a map $F : \mathcal{C} \to \mathbf{Grpd}$ such that every commutative triangle of morphisms in $\mathcal{C}$ is sent to a triangle functors of groupoids that commutes up to natural isomorphism (see diagram below).

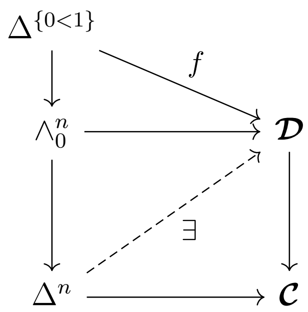


A functor is defined on objects and morphisms up to equality, but we do not necessarily want to choose a representative object or morphism in each morphism class, thus the map F is not a functor. In order to construct such a map F it was noticed by Grothendieck that one may instead construct a functor $p : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ as in the following pullback diagram with fibres $D_I \in \mathbf{Grpd}$ for $I \in \boldsymbol{\mathcal{C}}$ (see *Grothendeick construction* [18, p. 55-57]).



Remark 3.2.6. Writing down an $(\infty, 1)$-functor requires specifying an infinite amount of data and coherence conditions on higher morphisms. The following analogous construction for ∞-categories is comparatively painless, and absolutely essential in constructing $(\infty, 1)$-functors.

Definition 3.2.7. Suppose we have an inner fibration of simplicial sets $p : X \to Y$. Then a morphism $f \in X$ is a *p-cocartesian edge* if it satisfies the following commutative diagram for all $n \in \mathbb{N}$ [18, Rem. 2.4.1.4]:

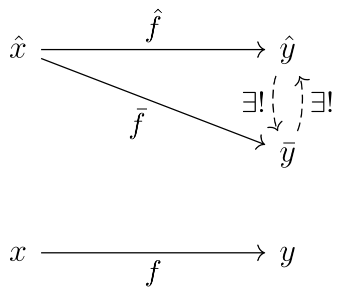


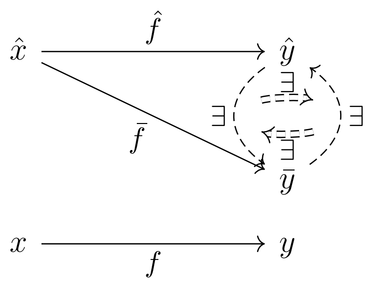
Such a morphism f is called a *cocartesian lift* of $p(f)$.

Remark 3.2.8. In the case where p is a functor of ordinary categories, for f to be a p-cocartesian edge we also require that the lift is unique [18, Prop. 1.1.2.2]. We shall call f an *ordinary* cocartesian edge.

Lemma 3.2.9. *In the ordinary categorical case, the cocartesian lift of a morphism is unique up to unique isomorphism. In the infinity categorical case the lift is unique up to equivalence of morphisms.*

Proof. The unique horn fillers complete the diagram below (left) of two cocartesian lifts of morphism f, giving us our desired isomorphism. Analogously, to obtain the equivalence in the infinity categorical case we must use all higher horn fillers of both cocartesian lifts. We show that there is no obstruction to finding maps in both directions at every level of higher morphism (see the diagram on the right for the 1 and 2-morphisms).





□

Definition 3.2.10. A *cocartesian fibration* [18, Def. 2.4.2.1] s an inner fibration of simplicial sets $p : X \to Y$ such that for all morphisms $f \in Y$ there exists a cocartesian lift of f. We shall refer to X as the *total category* and Y as the *base category*.

As the name suggests, there is the dual notion.

Definition 3.2.11. A *p-cartesian morphism* for morphism of simplicial sets $p : X \to Y$ is just a p^{op}-cocartesian morphism in $p^{\mathrm{op}} : X^{\mathrm{op}} \to Y^{\mathrm{op}}$. Likewise, for a *cartesian fibration* we stipulate that an inner fibration must have a *cartesian lift* for each morphism in the base category [18, Def. 2.4.2.1].

Definition 3.2.12. Cartesian fibrations p of ordinary categories define *Grothendieck fibrations* [24, Def. 2.2] such that $p : \boldsymbol{\mathcal{D}} \to \boldsymbol{\mathcal{C}}$ is a functor and for every morphism of $\boldsymbol{\mathcal{C}}$ there exists an ordinary cartesian lift.

Definition 3.2.13. Suppose $X : \Delta^{\mathrm{op}} \to \mathbf{Set}$ is a simplicial set. Consider the maps $\alpha_i : [1] \to [n]$ given by $\alpha_i(0) = i, \alpha_i(1) = i + 1$ and their dual maps α_i^{op}. When the functions $X(\alpha_i^{\mathrm{op}})$ give a map $X_n \to X_1 \times_{X_0} X_1 \times ... \times_{X_0} X_1$, we call this map is the n^{th}-*Segal map* (as adapted from [3, p. 406-407] for simplicial spaces).

Definition 3.2.14. The *Segal condition* [17, Prop. 4.4.1.11] on an inner fibration of simplicial sets $p : X \to \Delta^{\text{op}}$ specifies that all the Segal maps are equivalences.

Remark 3.2.15. Suppose X is the nerve of a category $X := N\boldsymbol{\mathcal{C}}$. The n-simplicies X_n are just the tuples of n composable morphisms: n elements of X_1 such that adjacent morphisms agree on their codomain and domain objects (0-simplices in X_0). Therefore, X satisfies the Segal condition. In fact, if a simplicial set satisfies the Segal condition then it must be equivalent to the nerve of some category (as first discussed in [9], and restated in [22, §2]).

Definition 3.2.16. A *monoidal category* is a cocartesian fibration of ordinary categories $p : X^{\otimes} \to \Delta^{\text{op}}$ satisfying the ordinary categorical Segal condition. A *monoidal ∞-category* is a cocartesian fibration

$$p : X^{\otimes} \to N\Delta^{\text{op}}$$

satisfying the Segal condition [17, p. 166-167]. Often the fibration is omitted and we call just the total category $X^{\otimes}$ a monoidal (∞-)category.

Example 3.2.17. The functor $\mathbf{Vect}^{\otimes} \xrightarrow{p} \Delta^{\text{op}}$ seen in Construction 2.2.2 is an instance of a general construction in [17, p. 166-167, (M1)]. Thus it is a cocartesion fibration of ordinary categories and indeed the Segal condition is satisfied. Again, we arrive at the conclusion that $\mathbf{Vect}^{\otimes}$ is a monoidal category. The p-cocartesian lift of a morphism $\phi \in \Delta^{\text{op}}$ is precisely the lift (2.1).

Now we are ready to begin tackling Barwick's theory of operads, and tie together the stories we have begun to introduce so far.

Chapter 4

Operator Categories (I)

4.1 Operator Categories

4.1.1. If operator categories are to generalise operads they had better have some key features. Namely, a collection of objects indexed by finite sets $\langle n \rangle \in \boldsymbol{\mathcal{F}}$, that correspond to n-ary operations. In order to compose operations as in operads, we might ask that given a map $\phi : J \to I \in \Phi$ between operations, we should to be able to take fibres over 'elements' of object I, which partitions J into $|I|$ many J_i-ary operations, like so $P(J) \times \prod_{i \in |I|} P(J_i) \to P(\sum_{i \in |I|} J_i),$. From this interpretation the definition of an operator category becomes very natural.

Definition 4.1.2. An *operator category* [1, Def. 1.2] is an essentially small category Φ with the following properties:

- there exists a terminal object $1 \in \Phi$;
- for all morphisms $\phi : J \to I \in \Phi$ the fibre $\{i\} \times_I J$ exists in Φ;
- for all $I, J \in \Phi$ the set $\mathrm{Hom}_\Phi(I, J)$ is finite.

How elegant that a theory of operads might be determined by a category with so few properties.

Remark 4.1.3. Every operator category Φ admits a functor $|\cdot|$ into $\boldsymbol{\mathcal{F}}$ such that $I \mapsto |I| := \mathrm{Hom}_\Phi(\mathbb{1}, I)$ [1, Ex. 1.11.2]. We will call $|I|$ the set of points of I. Then the fibre of $|\cdot|$ over $\langle n \rangle \in \boldsymbol{\mathcal{F}}$ is roughly the collection of n-ary operations.

Definition 4.1.4. A functor between operator categories is *admissible* if it preserves both the terminal object and the fibre construction [1, Def. 1.10]. Along with all natural isomorphisms, operator categories and admissible functors make up the 2-category Adm [1, Not. 1.13].

Continuing our comparison to operads it seems sensible to consider morphisms under which n-ary operations are mapped to n-ary operations. This gives us the following definition.

Definition 4.1.5. An *operator morphism* [1, Def. 1.10] is an admissible functor G between operator categories that induces a bijection on points, $|I| \mapsto |GI|$. Similarly to Adm we arrive at the 2-category Op of operator categories and operator morphisms.

Remark 4.1.6. Barwick's paper states the same definition except that the induced map on points is only required to be surjective. By [1, Prop.1.12] these are equivalent conditions.

Example 4.1.7. The terminal category $\mathbb{1}$ is an operator category [1, Ex. 1.4.1]. Trivially, for operator category Φ the unique functor $! : \Phi \to \mathbb{1}$ is admissible [1, Ex. 1.11.5].

Example 4.1.8. The points functor $| \cdot | : \Phi \to \boldsymbol{\mathcal{F}}$ is by definition an operator morphism [1, Ex. 1.11.2]. In fact, in the category of operator categories and operator morphisms the category $\boldsymbol{\mathcal{F}}$ is terminal [1, Prop. 1.14].

Remark 4.1.9. The category of finite graphs **Grph** with graph homomorphisms then the terminal object is a graph with a single vertex with a loop. The points of a graph G would then be its loops.

Example 4.1.10. If we consider finite simple graphs $\mathbf{Grph}_s$ and graph homomorphisms that allow contraction of edges. Then the points $|G|$ of a graph G in this operator category are its vertices. The embedding $\iota : \boldsymbol{\mathcal{F}} \to \mathbf{Grph}$ which sends $\langle n \rangle$ to a n-vertex graph with no edges, is an operator morphism.

4.2 Wreath Products

4.2.1. One key motivator for developing a theory of operads is to understand how to specify compatible algebraic structure from multiple operads in the same object. We pose the following problem: say we have an algebra $\mathbb{V}$ over the operad $\mathbb{E}_1$, how might one define a second monoidal product in $\mathbb{V}$ compatible with the first? We may take an $\mathbb{E}_1$-algebra in the category of $\mathbb{E}_1$-algebras as we saw in Example 2.1.4 when taking a monoid in the category **Mon**. As we will see in Section 5.4 this is the same as taking algebras over the operads $\mathbb{E}_2$. If we want to generalise this idea to yield a product of operads, then we might look for a product of operator categories $\Psi \wr \Phi$ such that algebras over $\Psi \wr \Phi$-operads are objects with Ψ-multiplications with compatible Φ-multiplications.

Definition 4.2.2. Given operator categories Ψ and Φ, the *wreath product* $\Psi \wr \Phi$ [1, Ex. 1.6] consists of:

- objects $(I, \{M_i\}_{i \in |I|})$ for $I \in \Phi$ and $M_i \in \Psi$;
- morphisms given by $(\phi : I \to J, \{\eta_i : M_i \to N_{\phi(i)}\}_{i \in |I|})$ for morphisms $\phi \in \Phi$ and $\eta_i \in \Psi$.

Proposition 4.2.3. *Given operator categories Ψ and Φ, their wreath product $\Psi \wr \Phi$ is also an operator category [1, Ex. 1.6].*

Proof. Suppose Φ and Ψ are operator categories then for their respective terminal objects 1_Φ and 1_Ψ $(1_\Phi, \{1_\Psi\})$ is terminal in $\Psi \wr \Phi$. For object $I^M := (I, \{M_i\}_{i \in |I|}) \in \Phi$ and morphism $J^N \to I^M \in \Psi \wr \Phi$, the fibre over i is $J_i^N := (J_i, \{N_j\}_{j \in J_i})$.

It is simple to check the combinatorics of sets of isomorphism classes to verify that $\Psi \wr \Phi$ is esssentially small. Likewise for the finiteness condition on hom sets. □

Example 4.2.4. The trivial category $\mathbb{1}$ is the unit of the wreath product— for operator category Φ we have $\mathbb{1} \wr \Phi = \Phi = \Phi \wr \mathbb{1}$.

Example 4.2.5. An important category in our story is the iterated wreath product of $\mathcal{O}$ with itself: $\mathcal{O}^{(n)} := \underbrace{\mathcal{O} \wr \ldots \wr \mathcal{O}}_{n\text{-fold product}}$.

Notation 4.2.6. For an operator category Φ where $i \in |I|$ is a point in $I \in \Phi$, it is usual to denote the morphism $1 \xrightarrow{i} I$ by $\{i\} \to I$. Similarly when keeping track of such a morphism we may often denote the fibre of $X \xrightarrow{p} N\Phi$ over $1 \in \Phi$ by $X_{\{i\}}$ instead of X_1.

Definition 4.2.7. For operator category Φ, a *coronal fibration* $X \xrightarrow{p} N\Phi$ [1, Def. 3.1] is a cartesian fibration such that the functors

$$\{X_I \to X_{\{i\}} | i \in |I|\}$$

give the fibre X_I as a product of the fibres $X_{\{i\}}$.

Remark 4.2.8. In the above definition, the Segal condition that $X_I \simeq \prod_{i \in |I|} X_i$ allows us to think of objects in X as $|I|$-tuples of objects in X_1 the fibre over the terminal object. These are analogous to objects $(J, (x_i)_{i \in |I|})$ in the wreath product $X_1 \wr \Phi$.

Proposition 4.2.9. *The inner fibration $(Op^{\wr})^{op} \to N\Delta^{op}$ is a monoidal ∞-category.*

Proof. See the proof of [1, Prop.3.9] for full detail. □

We will give instead a sketch and some key conceptual details.

Construction 4.2.10. This proof [1, Not. 3.8] requires a technical construction of a simplicial set over $N\Delta$ denoted $E(\mathbf{Adm})$, which allows us to conclude that $E(\mathbf{Adm})$ is a cartesian fibration. The inner fibration $\mathbf{Op}^{\wr} \to N\Delta$ is then specified as a subcategory $\mathbf{Op}^{\wr} \subseteq E(\mathbf{Adm})$.

Roughly this subcategory comprises the following data.

- The collection of objects (m, X) consisting of an integer $m \geq 0$ and a functor $X : (\Delta^m)^{\mathrm{op}} \to \mathbf{Adm}$. The image of X on Δ^m is a chain of operator categories and admissible functors $X_m \to X_{m-1} \to ... \to X_0$ such that for all $1 \leq i \leq m$ the nerve of $X_i \to X_{i-1}$ is a coronal fibration and $X_0 \simeq \mathbb{1}$ the trivial category.

- Morphisms in $E(\mathbf{Adm})$ are tuples $(\phi, g) : (Y, n) \to (X, m)$ for $\phi : [n] \to [m] \in \Delta$ and $g : Y \to \phi^* X \in \mathbf{Fun}((\Delta^n)^{\mathrm{op}}, \mathbf{Adm})$. This gives us the ∞-categorical data of a collection of morphisms $Y_i \to X_{\phi(i)}$. The morphisms in $\mathbf{Op}^{\wr}$ are those (ϕ, g) for which each $Y_i \to X_{\phi(i)}$ is an operator morphism.

Construction 4.2.11. It is intuitive that we should construct iterated wreath products of operator categories via chains of coronal fibrations. In diagram 4.1

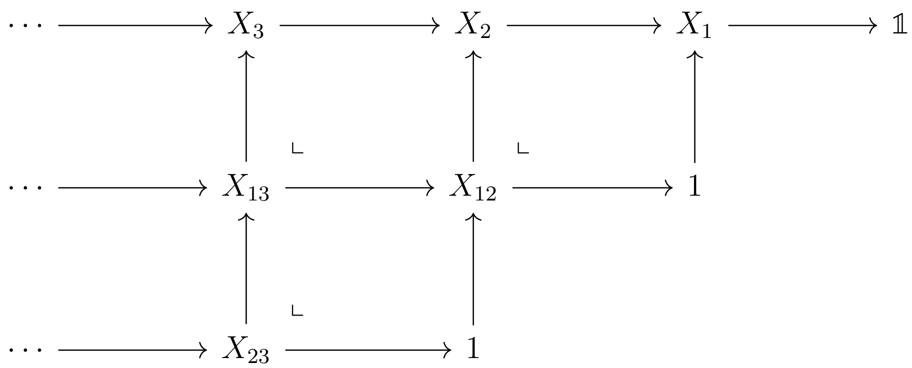


Figure 4.1: Grid of pullbacks of admissible functors of operator categories.

we have that all vertical arrows are operator morphisms and taking the nerve of each horisontal arrow gives us a coronal fibrations. By [1, Lem. 3.5] we may interpret this commutative diagram of pullbacks as giving the data of the following equivalences:

- from the top row $X_2 \simeq X_{12} \wr X_1$, $X_3 \simeq X_{13} \wr X_1$;

- from the second row $X_{13} \simeq X_{23} \wr X_{12}$ and so on.

This is the intuition for how the cocartesian fibration $(Op^{\wr})^{\mathrm{op}} \to N\Delta^{\mathrm{op}}$ $[n] \in \Delta$ allows us to recover the wreath product of n operator categories. In general, pullbacks allow us to arrive at the equivalence $X_n \simeq X_{(n-1)n} \wr \ldots \wr X_{12} \wr X_1$.

4.3 Perfect Operator Categories

There is a technical property of some operator categories which we shall exploit in order to have an elegant definition of a Φ-operad for an operator category Φ. All the categories we care about in the scope of this project satisfy this property.

Definition 4.3.1. The category of points of objects in Φ we will call Φ^{cons}. Its morphisms are pullback squares in Φ as in the diagram below. We define the *point classifier* of Φ to be the terminal object (T, t) in Φ^{cons} [1, Def. 4.1].

$$\begin{array}{ccc} \{i\} & \longrightarrow & I \\ \Vert & \lrcorner & \downarrow \\ \{j\} & \longrightarrow & J. \end{array}$$

We distinguish t as the *special point* of T.

Example 4.3.2. We may visualise an arbitrary object of $\mathcal{O}^{(2)}$ and its point classifier as in the diagrams below:

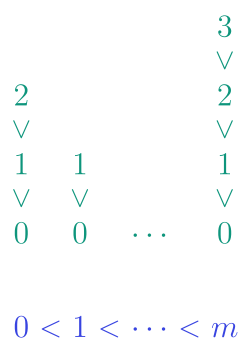


Figure 4.2: Object $([m], ([2], [1], ..., [3]))$ in $\mathcal{O}^{(2)}$.

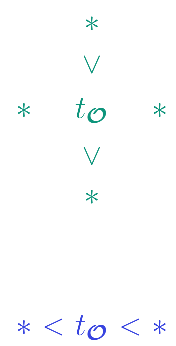


Figure 4.3: Point classifier $T_{\mathcal{O}^{(2)}} = (T_{\mathcal{O}}, (*, T_{\mathcal{O}}, *))$ in $\mathcal{O}^{(2)}$.

Imagine that the point classifier of $\mathcal{O}^{(n)}$ takes the origin in $\mathbb{R}^n$ and adds a point in the positive and negative direction along each axis (see Figure 4.4).

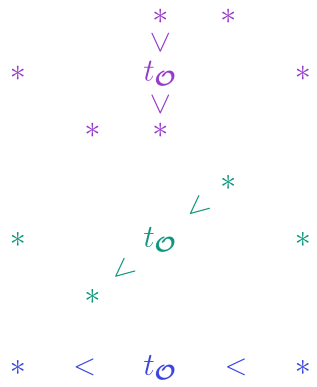
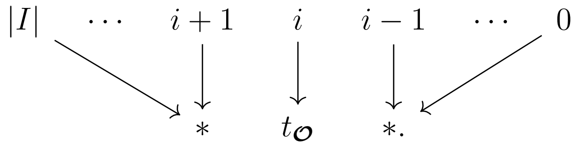


Figure 4.4: The point classifier $T_{\mathcal{O}^{(3)}}$.

Definition 4.3.3. For the monad T of a perfect operator category let the unique morphism $\chi_i : (I, i) \to (T, t)$ be called the *classifying morphism* of i [1, Def. 4.3].

Example 4.3.4. The point classifier for $\mathcal{O}$ is $(T_{\mathcal{O}}, t_{\mathcal{O}}) := (\{* < t_{\mathcal{O}} < *\}, t_{\mathcal{O}})$ and the classifying morphism $\chi_i : (I, i) \to (T_{\mathcal{O}}, t_{\mathcal{O}})$ is



Definition 4.3.5. The *special fibre functor* fib : $\Phi_{/T_\Phi} \to \Phi$ is given by $I \mapsto I_{t_\Phi}$, the fibre over special point $t_\Phi \in T_\Phi$ [1, Def. 4.3].

We are now ready to give the technical property we want our operator categories to satisfy.

Definition 4.3.6. We shall call an operator category Φ *perfect* if it satisfies the following conditions [1, Def. 4.6]:

- the operator category Φ has a point classifier;
- the special fibre functor fib admits a right adjoint $E_\Phi : \Phi \to \Phi_{/T_\Phi}$.

Where it is clear which perfect operator category the monad T_Φ and right adjoint E_Φ belong to, we will drop the subscripts.

Construction 4.3.7. The right adjoint E for perfect operator category Φ sends I to EI where EI denotes the object TI along with a *structure morphism* $e_I =$

$T(I \to 1) : TI \to T$ where the fibre of e_I over $t \in T$ is isomorphic to I [1, Not. 4.7].

Proposition 4.3.8. *The wreath product of perfect operator categories $\Psi \wr \Phi$ is also a perfect operator category [1, Prop. 4.10].*

Proof. Suppose Ψ and Ψ are perfect operator categories with point classifiers T_Φ, T_Ψ and adjunctions $fib_\Phi \dashv E_\Phi$ and $fib_\Psi \dashv E_\Psi$. Then $\Psi \wr \Phi$ has point classifier:

$$(T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|}),\ S_\eta = \begin{cases} 1, \eta \neq t_\Phi \\ T_\Psi, \eta = t_\Phi. \end{cases}$$

The special point is $t_{\Psi \wr \Phi} := (t_\Phi, t_\Psi)$. The image of the right adjoint $E_{\Psi \wr \Phi}$ on $(I, \{M_i\}_{i \in |J|})$ is

$$(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|}),\ N_j = \begin{cases} T_\Psi M_j, j \in |I| \subseteq |T_\Phi I| \\ 1, \text{otherwise}, \end{cases}$$

with structural morphism $(T_\Phi I, \{N_j\}_{j \in |T_\Phi I|}) \to (T_\Phi, \{S_\eta\}_{\eta \in |T_\Phi|})$ given by $e_I : T_\Phi I \to T_\Phi$, $e_{M_j} : T_\Psi M_j \to T_\Psi$ for $j \in |I| \subseteq |T_\Phi I|$, and the identity on $1 \in \Psi$ for $j \notin |I| \subseteq |T_\Phi I|$. □

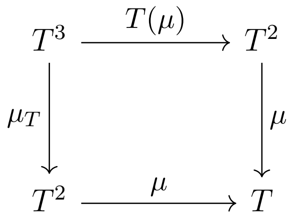
Chapter 5

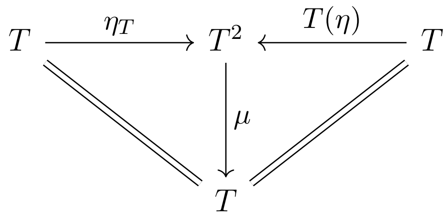
Operator Categories (II)

5.1 Monads

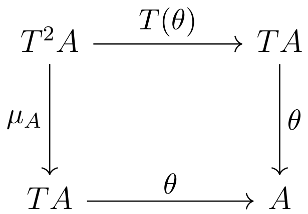
From perfect operator categories we are able to construct another type of object which is used to control algebraic structure.

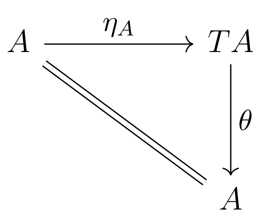
Definition 5.1.1. A *monad* [4, Def. 4.1.1] is an endofunctor $T : \mathcal{C} \to \mathcal{C}$ along with natural isomorphisms $\eta : id_{\mathcal{C}} \to T$, the *unit*, and $\mu : T^2 \to T$, the *multiplication*: which satisfy the following associativity and unitality commutative diagrams.





Definition 5.1.2. An *algebra over a monad* [4, Def. 4.1.2] $T : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}}$ is an object $A \in \boldsymbol{\mathcal{C}}$, along with a morphism $TA \xrightarrow{\theta} A$: which satisfies the following commutative diagrams.





The collection of all algebras over monad T form a category $\mathbf{Alg}_T$, where morphisms $m : (A, \theta_A) \to (B, \theta_B)$ are commutative squares

$$\begin{array}{ccc} TA & \xrightarrow{T(m)} & TB \\ {\scriptstyle \theta_A}\downarrow & & \downarrow{\scriptstyle \theta_B} \\ A & \xrightarrow{m} & B \end{array}$$

for $m \in \boldsymbol{\mathcal{C}}$.

Construction 5.1.3. The forgetful functor $U : \mathbf{Alg}_T \to \boldsymbol{\mathcal{C}}$ admits a left adjoint [4, Prop. 4.1.4]:

$$\boldsymbol{\mathcal{C}} \underset{U}{\overset{\text{Free}}{\rightleftarrows}} \mathbf{Alg}(T) \qquad (\text{Free} \dashv U)$$

Here Free carries $A \in \boldsymbol{\mathcal{C}}$ to the *free algebra* (TA, μ_A) generated by A [4, Def. 4.1.5]. Let $\mathbf{Free}(T)$ be the essential image of Free in $\mathbf{Alg}_T$.

Definition 5.1.4. The *Kleisli category* of monad T [4, Prop. 4.1.6] is a category $\boldsymbol{\mathcal{C}}_T$ in which:

- the objects of $\boldsymbol{\mathcal{C}}_T$ are B_T for all $B \in \mathcal{C}$;
- the morphisms are $f_T : A_T \to B_T$ for all $f : A \to TB \in \boldsymbol{\mathcal{C}}$;
- the identity on object A_T $id_{A_T} := \eta_A$;
- composition of $f_T : A_T \to B_T$, $g_T : B_T \to C_T$ in $\boldsymbol{\mathcal{C}}_T$ we define by $g_T \circ f_T := \mu_C \circ T(g) \circ f$.

The Kleisli category of a monad T has an important relationship to the category of algebras over T, as we will now show.

Construction 5.1.5. Let F be the functor $\boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}}_T$ which:

- is the identity on objects;
- sends morphisms $f \in \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(A, B)$ to $\bar{f} : A \to TB$ via the unit of T.

Lemma 5.1.6. *There exists a functor $\mathcal{C}_T \to \mathbf{Alg}_T$ such that the following diagram commutes.*

$$\begin{array}{ccc} & & \mathbf{Alg}_T \\ & \overset{\mathit{Free}}{\nearrow} & \uparrow \exists \\ \mathcal{C} & & \\ & \underset{F}{\searrow} & \\ & & \mathcal{C}_T \end{array} \tag{5.1}$$

Proof. Let $G : \mathcal{C}_T \to \mathbf{Alg}_T$ be the functor which:

- sends $A \in \mathcal{C}_T$ to the free algebra generated by A;
- sends morphisms $f \in \mathrm{Hom}_{\mathcal{C}_T}(A, B)$ to $\mathbf{Alg}_T$ along the composite with the free-forgetful adjunction:

$$\mathrm{Hom}_{\mathcal{C}_T}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, TB) \cong \mathrm{Hom}_{\mathbf{Alg}_T}(TA, TB).$$

Then G satisfies the commuting diagram (5.1). □

Lemma 5.1.7. *The Kleisli category $\mathcal{C}_T$ is equivalent to **Free**(T) the category of free T-algebras.*

Proof. The functor G in the proof of Lemma 5.1.6 is clearly essentially surjective on **Free**(T) and is fully faithful by the isomorphism on maps given by the adjunction. Hence, G is an equivalence of categories. □

5.2 Colocalisation

5.2.1. A canonical adjunction associated with a perfect operator category Φ, as introduced in the following section, asserts a nice relationship between Φ and $\Phi_{/T}$ for point classifier $T \in \Phi$. We shall introduce the relevant background to understand it here.

Definition 5.2.2. Let S denote a collection of morphisms in a category $\mathcal{C}$, then an object $A \in \mathcal{C}$ is called *S-local* with if for all morphisms $f \in S$ the function $\mathrm{Hom}(f, A)$ is an isomorphism. (Adapted from [11, Def. 3.1.4.(a)] for model categories).

In other words A 'thinks' that all $f \in S$ are isomorphisms. Where there is no ambiguity we shall just refer to such objects as *local*.

Definition 5.2.3. Suppose we have an adjunction:

$$\boldsymbol{\mathcal{C}} \underset{i}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \boldsymbol{\mathcal{D}}\ ,$$

where i is a fully faithful functor. Then $\boldsymbol{\mathcal{D}}$ is a *localisation* of $\boldsymbol{\mathcal{C}}$ (originally referred to in the literature as the *category of fractions* of $\boldsymbol{\mathcal{C}}$) [7, Prop. 1.3].

Definition 5.2.4. Suppose that the adjunction $L \dashv i$ exhibits $\boldsymbol{\mathcal{C}}$ as a localisation of $\boldsymbol{\mathcal{D}}$. Let the category $\boldsymbol{\mathcal{W}} = \{f \in \boldsymbol{\mathcal{C}} \mid Lf \text{ is an isomorphism}\}$ be the category of *local isomorphisms* of localisation $L \dashv i$. (Adapted from [11, Def. 3.1.4.(b)] for model categories).

Lemma 5.2.5. *Suppose $\boldsymbol{\mathcal{D}}$ is a localisation of $\boldsymbol{\mathcal{C}}$ via the adjunction $L \dashv i$. Now consider the wide subcategory $\boldsymbol{\mathcal{W}}$ of local isomorphisms, then the category $\boldsymbol{\mathcal{D}}$ is equivalent to the full subcategory of W-local objects in $\boldsymbol{\mathcal{C}}$.*

Proof. Firstly, we show that for all $x \in \boldsymbol{\mathcal{D}}$ we have that $i(x)$ is local. It is clear for all $f : B \to A$ the following diagram commutes by naturality of the adjunction.

$$\begin{array}{ccc} \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(A, ix) & \xrightarrow{\cong} & \mathrm{Hom}_{\boldsymbol{\mathcal{D}}}(LA, x) \\ {\scriptstyle \mathrm{Hom}(f, ix)}\big\downarrow & & {\scriptstyle \wr\|}\big\downarrow{\scriptstyle \mathrm{Hom}_{\boldsymbol{\mathcal{D}}}(Lf, x)} \\ \mathrm{Hom}_{\boldsymbol{\mathcal{C}}}(B, ix) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\boldsymbol{\mathcal{D}}}(LB, x). \end{array}$$

and so $Hom(f, i(x))$ is an isomorphism for all f and hence local.

Secondly, we have a local object $y \in \boldsymbol{\mathcal{C}}$ and we want to show that there exists an $x \in \boldsymbol{\mathcal{D}}$ such that $y \cong ix$, namely $x = Ly$. We use that the counit of the adjunction $\eta : id_{\boldsymbol{\mathcal{C}}} \to i \circ L$ is an isomorphism. By the triangle identities for the adjunction we get that for local object y that $L(\eta_y)$ is an isomorphism, and so η_y is a local isomorphism.

Given a local isomorphism $f : x \to y$, if x, y are local objects then f is an isomorphism. Given the full subcategory of local objects $\boldsymbol{\mathcal{C}}^{\boldsymbol{\mathcal{W}}}$ then $\mathrm{Hom}(f, -) : \boldsymbol{\mathcal{C}}^{\boldsymbol{\mathcal{W}}} \to \mathbf{Set}$ is a natural isomorphism and so f is an isomorphism. Therefore, if y is as local object then $y \xrightarrow{\eta_y} iLy$ is an isomorphism, and D is equivalent to $\boldsymbol{\mathcal{C}}^{\boldsymbol{\mathcal{W}}}$.

□

Definition 5.2.6. Dually, suppose we have an adjunction:

$$\boldsymbol{\mathcal{D}} \underset{R}{\overset{i}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} \boldsymbol{\mathcal{C}} \,,$$

where i is a fully faithful functor. Then $\boldsymbol{\mathcal{C}}$ is *colocalisation* of $\boldsymbol{\mathcal{D}}$.

5.3 Canonical Monad on a Perfect Operator Category (Chapter 5 [1])

We will construct a monad for any perfect operator category Φ with the endofunctor $T := U \circ E_\Phi$ where E_Φ is the right adjoint to special fibre functor fib and U is the forgetful functor which forgets the structural morphism e_I, sending $EI \in \Phi_{/T}$ to $TI \in \Phi$. The unit of the monad is simple to state so this section will be concerned primarily with the construction of the multiplication $\mu : T^2 \to T$.

Construction 5.3.1. We let unit of the monad T, $\iota : id_\Phi \to T$, be given by components $\iota_I : I \to TI$ which embeds I in TI (constructed in proof of [1, Lem. 5.2], and stated as the unit in [1, Thm. 5.10]).

Example 5.3.2. The example you should keep in mind for all morphism constructions is for the perfect operator category $\boldsymbol{\mathcal{O}}$:

$$\begin{array}{ccccccc} TI & \{* & |I| & \cdots & 0 & *\} \\ \big\downarrow e_I & & \searrow & \searrow & \swarrow & \swarrow \\ T & & \{* & t_{\mathcal{O}} & *\} & \end{array} \qquad \begin{array}{ccccccc} I & & \{0 & \cdots & i & \cdots & |I|\} \\ \big\downarrow \iota_I & & \downarrow & & \downarrow & & \downarrow \\ TI & \{* & 0 & \cdots & i & \cdots & |I| & *\} \end{array}$$

It is clear that $I \in \boldsymbol{\mathcal{O}}$ also satisfies the universal property of the following pullback square:

$$\begin{array}{ccc} \mathrm{fib} \circ E(I) = (TI)_t & \longrightarrow & TI \\ \big\downarrow & \ulcorner & \big\downarrow e_I \\ \{t\} & \hookrightarrow & T \end{array}$$

The unique isomorphism $I \to \mathrm{fib} \circ E(I)$ is necessarily the I^{th} component of the counit of the adjunction $\mathrm{fib} \dashv E$.

Construction 5.3.3. We will now construct the multiplication of monad $T := U \circ E$.

Define $\chi_{t,!} : \Phi_{/TT} \to \Phi_{/T}$ to be the functor given by composition with classifying morphism χ_t (Def. 4.3.3). We will use the following natural transformations to construct multiplication μ of monad T [1, Def. 5.8]:

- the counit $\kappa : \text{fib} \circ E \to id_\Phi$ with I^{th} component $\kappa_I : (TI)_t \to I$;
- natural transformation $\sigma : \chi_{i,!} \circ E_{/T} \to E \circ \text{fib}$ by components σ_I as in the following commutative diagram.

$$\begin{array}{ccc} TI & \xrightarrow{\sigma_I} & T(I_t) \\ {\scriptstyle T(\phi)}\downarrow & & \downarrow{\scriptstyle e_{I_t}} \\ TT & \xrightarrow[\chi_t]{} & T \end{array}$$

Here I_t is the fibre of $I \xrightarrow{\phi} T$.

We put them together to express the I^{th}-component of μ is shown as the top edge of the following commutative diagram [1, Diag. 5.9.1].

$$\begin{array}{ccccc} T^2 I & \xrightarrow{\sigma_{TI}} & T((TI)_t) & \xrightarrow[\cong]{T(\kappa_I)} & TI \\ {\scriptstyle T(e_I)}\downarrow & & & & \downarrow{\scriptstyle e_I} \\ TT & & \xrightarrow[\chi_t]{} & & T \end{array}$$

Example 5.3.4. The I^{th} component of multiplication $\mu_I : T^2 I \to TI$ for perfect operator category $\mathcal{O}$ is the morphism determined by sending only points in $I \subseteq T^2 I$ to points in $I \subseteq TI$, mapping them in isomorphically. It is obvious that the unit and associativity axioms of a monad in Definition 5.1.1 are satisfied.

Theorem 5.3.5. The endofunctor $T_\Phi := U \circ E_\Phi$ on perfect operator category Φ with the multiplication μ (Construction 5.3.3) and identity ι (Construction 5.3.1) is a monad on Φ [1, Thm. 5.10].

Proof. See [1, Appendix. B]. □

Example 5.3.6. The canonical monad T on $\mathcal{F}$ is the *partial map monad* [1, Ex. 5.11], $T\langle n \rangle = \langle n \rangle_+$. Maps $T(F) : \langle n \rangle_+ \to \langle m \rangle_+$ are can be thought of as partial maps out of $\langle n \rangle$ where we do not consider points in $\langle n \rangle$ which are mapped to $+ \in \langle m \rangle_+$ by $T(f)$.

Recall in Example 1.3.5 we looked at the symmetric monoidal category $\mathbf{Vect}^{\otimes}$ where morphisms were defined in terms of partial maps. We may define these maps instead in terms of maps in $\mathcal{F}_+$, the category of pointed finite sets. This is another hint at pieces of the story coming together as we develop this theory of operads.

Example 5.3.7. The canonical monad of $\mathcal{O}^{(n)}$ is the endofunctor which adds a new maximum and minimum to each linearly ordered set in $\mathcal{O}$ [1, p. 1895].

Due to the following Lemma, we know that the counit of the adjunction $fib_\Phi \dashv E_\Phi$ exhibits Φ as a colocalisation of $\Phi_{/T}$ [1, Lem. 5.2].

Lemma 5.3.8. *For perfect operator category Φ the functor E_Φ is fully faithful.*

Proof. Define functor $p_t : \Phi \to \Phi_{/T}$ as sending $II \to t \to T$. Clearly this gives us the adjunction $p_t \dashv \text{fib}$ where the unit is a natural isomorphism whose components are $I \cong (p_t(I))_t$. Thus we have a chain of adjoints $p_t \dashv \text{fib} \dashv E$. We may compose adjunctions and get $\text{fib}(p_t) \dashv \text{fib}(E)$. The functor p_t is obviously fully faithful and so $\text{fib}(p_t) = id_\Phi$. The identity functor is left adjoint to itself, by uniqueness of adjoints $\text{fib}(E) = id_\Phi$, and so E is fully faithful. □

Example 5.3.9. The point classifier in **Grph** is a graph with two vertices $*$ and $t_{\mathbf{Grph}}$ with one edge between them. Looking for the right adjoint E becomes a lot easier using Lemma 5.3.8 above, in that for a graph G we want that $\text{fib} \circ E(G) \cong G$. The counit of the adjunction is an isomorphism. Therefore, we see that TG is obtained by adding a vertex $*$ and for every existing vertex in G an edge connecting it to $*$ (see Figure 5.1).

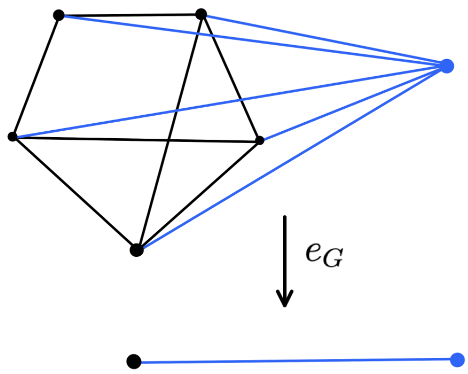


Figure 5.1: Image of of 5 vertex graph G (in black) under E with structural morphism $T_{\mathbf{Grph}}G \to T_{\mathbf{Grph}}$.

Remark 5.3.10. As suggested by the previous examples, the canonical monad on a perfect operator category can be thought of as giving us a universal way of adding points to objects [1, p. 1897]. It is universal in the sense that it adds to an object as few points in as many directions as possible. This intuition is displayed clearly in Figure 4.4 of the point classifier of $\boldsymbol{\mathcal{O}}^{(3)}$.

5.3.11. In the next stage of our story we follow observations 5.13 through 5.15 in Barwick's paper [1]. We summarise the construction of natural transformation α_F for any admissible functor of perfect operator categories $F : \Psi \to \Phi$. The assertion in [1, Thm. 5.18] we have that $(F, \alpha_F) : (\Phi, T_\Phi) \to (\Psi, T_\Psi)$ is the morphism of monads given in the following definition.

Definition 5.3.12. For monads $T_{\boldsymbol{\mathcal{C}}} : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{C}}$ and $T_{\boldsymbol{\mathcal{D}}}$ a *colax morphism of monads* [1, 5.17] is a functor $F : \boldsymbol{\mathcal{C}} \to \boldsymbol{\mathcal{D}}$ and a natural transformation $\alpha_F : T_{\boldsymbol{\mathcal{C}}} F \to F T_{\boldsymbol{\mathcal{D}}}$ satisfying some properties [1, 5.13].

This induced morphism of monads will be useful to us in the next chapter in proving the main result about the Kleisli category of the monad on $\boldsymbol{\mathcal{O}}^{(n)}$.

Notation 5.3.13. For any morphism $f : X \to Y$ let the function $f_! : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ for any A be composition with f.

Construction 5.3.14. Admissible functor $F : \Psi \to \Phi$ induces the functor $F_{/T_\Psi} : (\Psi_{/T_\Psi}) \to (\Phi_{/FT_\Psi})$. We remind the reader that the classifying morphism $\chi_{F(t_\Psi)} : (FT_\Psi, F(t_\Psi)) \to (T_\Phi, t_\Phi)$ is the unique morphism in Φ that sends only $F(t_\Psi)$ to $t_\Phi \in T_\Phi$. Finally, we let $F_{/T} := \chi_{F(t_\Psi),!} \circ F_{/T_\Psi}$.

Example 5.3.15. Let us make explicit such a functor $F_{/T}$ in the case where $F = |\cdot| : \boldsymbol{\mathcal{O}} \to \boldsymbol{\mathcal{F}}$. The functor $F_{/T} : (\boldsymbol{\mathcal{O}}_{/T_{\boldsymbol{\mathcal{O}}}}) \to (\boldsymbol{\mathcal{F}}_{/T_{\boldsymbol{\mathcal{F}}}})$ is given on objects and morphisms as seen below:

$$
\begin{array}{ccc}
\begin{array}{ccc}
[n] & \longrightarrow & [m] \\
\downarrow & \swarrow & \\
T_{\boldsymbol{\mathcal{O}}} \cong [2] & &
\end{array}
& \longmapsto &
\begin{array}{ccc}
\langle n+1 \rangle & \longrightarrow & \langle m+1 \rangle \\
\downarrow & & \swarrow \\
\langle T_{\boldsymbol{\mathcal{O}}} \rangle & = & \langle T_{\boldsymbol{\mathcal{O}}} \rangle \\
\downarrow \chi_{t_{\boldsymbol{\mathcal{O}}}} & \swarrow & \\
T_{\boldsymbol{\mathcal{F}}} & &
\end{array}
\end{array}
$$

We shall give two simple but clarifying points.

(1) We write $\langle T_{\boldsymbol{\mathcal{O}}} \rangle$ for $|T_{\boldsymbol{\mathcal{O}}}| \cong \langle 3 \rangle$ and so the leftmost triangle and the triangle on the top right (the image under $F_{/T_{\boldsymbol{\mathcal{O}}}}$) are identical as set maps;

(2) if the point classifier $T_{\mathcal{F}}$ consists of the special point $t_{\mathcal{F}}$ and the *bin* point $*$ then the classifying morphism $\chi_{t_{\mathcal{O}}} : (\langle T_{\mathcal{O}}\rangle, t_{\mathcal{O}}) \to (T_{\mathcal{F}} \cong \langle 2\rangle, t_{\mathcal{F}})$ throws every point except $t_{\mathcal{O}}$ in the bin.

Construction 5.3.16. Let the natural transformation $\alpha_F : F_{/T} \circ E_\Psi \to E_\Phi \circ F$ be determined by the components $\alpha_{F,I}$ that satisfy the following unique commutative square [1, Prop. 5.16]:

$$\begin{array}{ccc} FT_\Psi I & \xrightarrow{\alpha_{F,I}} & T_\Phi FI \\ \big\downarrow & & \big\downarrow \\ FT_\Psi & \xrightarrow[\chi_{F(t_\Psi)}]{} & T_\Phi, \end{array}$$

whose special fibre is the square

$$\begin{array}{ccc} FI & \overset{\alpha_{F,I}}{=\!=\!=} & FI \\ \big\downarrow & & \big\downarrow \\ F\{t_\Psi\} & \underset{\chi_{F(t_\Psi)}}{=\!=\!=} & \{t_\Phi\}. \end{array}$$

Example 5.3.17. Again for the admissible functor $F = |\cdot|$, we shall make the component of natural transformation $\alpha_F : F_{/T} \circ E_\Psi \to E_\Phi \circ F$ at $[m]$ explicit.

$$\alpha_{F,[m]} : \quad F_{/T} \circ E_{\mathcal{O}}([m]) \longrightarrow E_{\mathcal{F}} \circ F([m])$$

$$F_{/T}\left(\begin{array}{c} T_{\mathcal{O}}[m] \\ \big\downarrow e_{[m]} \\ T_{\mathcal{O}} \end{array}\right) = \begin{array}{c} \langle T_{\mathcal{O}}[m]\rangle \\ \big\downarrow \\ \langle T_{\mathcal{O}}\rangle \\ \big\downarrow \\ T_{\mathcal{F}} \end{array} \qquad \begin{array}{c} T_{\mathcal{F}}\langle m+1\rangle \\ \big\downarrow e_{\langle m+1\rangle} \\ T_{\mathcal{F}} \end{array} = E_{\mathcal{F}}(\langle m+1\rangle)$$

In the diagram above $e_{[m]}$ and $e_{\langle m+1\rangle}$ are the structural morphisms in the image of $E_{\mathcal{O}}$ and E_Ψ respectively. The component $\alpha_{F,[m]}$ consists of the set map that sends $[m] \subseteq T_{\mathcal{O}}[m]$ in $\langle T_{\mathcal{O}}[m]\rangle$ isomorphically to $\langle m+1\rangle \subseteq T_{\mathcal{F}}\langle m+1\rangle$, and the remaining points to the bin point $*$.

Theorem 5.3.18. An admissible functor $F : \Psi \to \Phi$ induces a colax morphism of monads $(F, \alpha_F) : (\Psi, T_\Psi) \to (\Phi, T_\Phi)$ for natural transformation α_F in Construction 5.3.16 [1, Thm. 5.18].

Proof. See appendix C [1]. □

5.4 Leinster Categories

5.4.1. In this chapter we finally introduce a generalisation of what Δ^{op} is to the operator category $\mathcal{O}$. In Chapter 2 we saw how we were able to define a monoidal structure (what we will later call $\mathcal{O}$-monoidal) via a map into Δ^{op}. This is a relationship we would like to understand and an incredible useful construction we would like to extend. And so we begin our journey to defining Φ-monoidal structures for arbitrary perfect operator category Φ.

Definition 5.4.2. Let Φ be a perfect operator category then the *Leinster* category $\Lambda(\Phi)$ is the Kleisli category of the canonical monad T_Φ [1, Def. 6.1]. Explicitly, the objects of $\Lambda(\Phi)$ are the objects of Φ and the morphisms $\text{Hom}_{\Lambda\Phi}(I, J) := \text{Hom}_\Phi(I, T_\Phi J)$.

Proposition 5.4.3. *The Leinster category $\Lambda(\mathcal{O})$ is Δ^{op} [1, Ex. 6.6] under the equivalence $Hom_{\Lambda(\mathcal{O})}(-, \varnothing) : \Lambda(\mathcal{O})^{op} \to \Delta$.*

Proof. Let $\text{Hom}_{\Lambda(\mathcal{O})}([0], \varnothing) = \text{Hom}_{\mathcal{O}}([0], T_{\mathcal{O}}(\varnothing)) = \{\bot, \top\}$. Explicitly,

- map $\top$ sends $t_{\mathcal{O}} \mapsto 0 \in T_{\mathcal{O}}(\varnothing)$;
- map $\bot$ sends $t_{\mathcal{O}} \mapsto 1 \in T_{\mathcal{O}}(\varnothing)$.

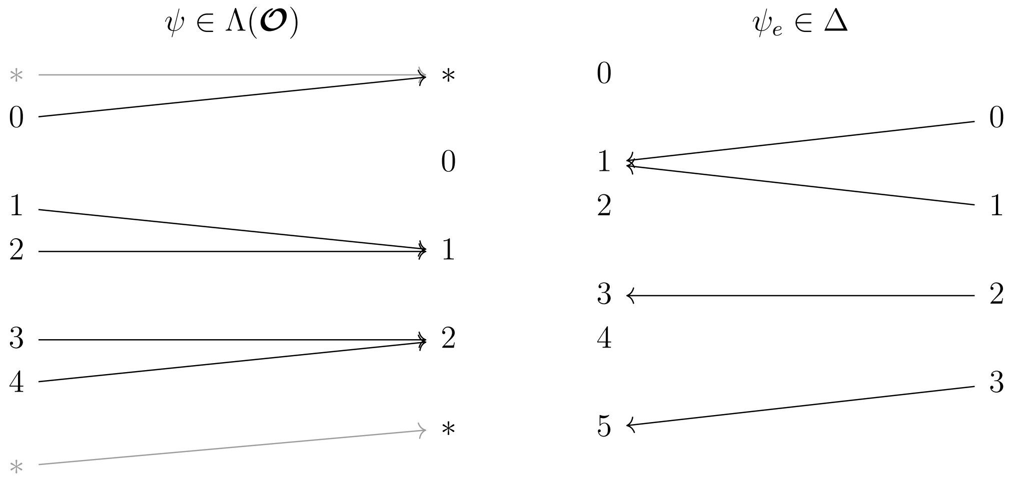
Let $c_I := (\bot_!, \top_!) : \text{Hom}_{\Lambda(\mathcal{O})}(I, [0]) \to \text{Hom}_{\Lambda(\mathcal{O})}(I, \varnothing) \times \text{Hom}_{\Lambda(\mathcal{O})}(I, \varnothing)$ then $c_I(\text{Hom}_{\Lambda(\mathcal{O})}(I, [0]))$ gives a linear ordering on $\text{Hom}_{\Lambda(\mathcal{O})}(I, \varnothing)$. Therefore, $\text{Hom}_{\Lambda(\mathcal{O})}(-, \varnothing) : \Lambda(\mathcal{O})^{\text{op}} \to \Delta$ gives a functor $[n] \mapsto \mathcal{O}([n], [1]) = [n+1]$ [1, Ex. 6.6]. □

Example 5.4.4. Here we will illustrate how $c_I(\text{Hom}_{\Lambda(\mathcal{O})}(I, [0]))$ gives an ordering on $\text{Hom}_{\Lambda(\mathcal{O})}(I, \varnothing)$ in the proof above.

- Let $i \in \text{Hom}_{\mathcal{O}}(I, T(\varnothing))$ be the morphism determined by $j \mapsto 0$ for $j \leq i$;
- let $f \in \text{Hom}_{\Lambda(\mathcal{O})}(I, [0])$ be the map determined by $j \mapsto 0$ for $i < j \leq i + k$;

then $c_I(f) = (i, i + k)$ giving that $i \leq i + k$.

Remark 5.4.5. The functor $\mathrm{Hom}_{\Lambda(\mathcal{O})}(-,\varnothing) : \Lambda(\mathcal{O})^{\mathrm{op}} \to \Delta$ can be understood on morphisms $\phi_T : I \to J \in \Lambda(\mathcal{O})$ by considering the end point preserving map $TI \to TJ \in \mathcal{O}$ that agrees with $\phi : I \to TJ$ on I.



Intuitively, the map $\psi_e \in \Delta$ corresponds to the dual map on edges induced by $\psi \in \Lambda(\mathcal{O})$. We notice from the diagram above that something similar is true in constructing ψ_e from ψ.

Proposition 5.4.6. *The Leinster category $\Lambda(\mathcal{F})$ is the category of pointed finite sets $\Gamma^{op} = \mathcal{F}_+$ [1, Ex. 6.5].*

Proof. Consider the monad T of which $\Lambda(\mathcal{F})$ is the Kleisli category. Define the functor $F : \Lambda(\mathcal{F}) \to \Gamma^{\mathrm{op}}$ where $I \mapsto TI$ and morphisms $\phi \in \mathrm{Hom}_{\Lambda(\mathcal{F})}(I,J) = \mathrm{Hom}_{\mathcal{F}}(I,TJ)$ are sent to point preserving morphisms $TI \xrightarrow{\phi_*} TJ$ such that $\phi = \iota_I \circ \phi_*$. Here ι_I is the embedding of I into TI, the I^{th} component of the unit of T. Clearly F is an equivalence of categories. □

Construction 5.4.7. An admissible functor of perfect operator categories $F : \Psi \to \Phi$ induces a functor $\Lambda(F) : \Lambda(\Psi) \to \Lambda(\Phi)$. This functor agrees with F on objects. On morphisms the functor agrees with the following composite:

$$\mathrm{Hom}_\Psi(I, T_\Psi J) \to \mathrm{Hom}_\Phi(FI, FT_\Psi J) \xrightarrow{\alpha_{F,*}} \mathrm{Hom}_\Phi(FI, T_\Phi FJ).$$

Here $\alpha_{F,*}$ takes a morphism in $\mathrm{Hom}_\Phi(FI, FT_\Psi J)$ and composes it with $\alpha_{F,J}$ (the J^{th} component of the natural transformation α_F from Construction 5.3.16).

Proposition 5.4.8. *An admissible functor $F : \Psi \to \Phi$ between perfect operator categories that is also a Grothendieck fibration induces a functor $\Lambda(F)$ which is also a Grothendieck fibration such that for any $I \in \Phi$ we have that $(\Lambda(\Psi))_I \simeq \Lambda(\Psi_I)$ [1, Prop. 6.7].*

Proof. Suppose F is a Grothendieck fibration. The $\Lambda(F)$-cartesian lift of $f_T : I \to J$ is the F-cartesian lift of $f : I \to TJ \in \Phi$, the unique $\wedge^2_2$-horn filler of f_T is obtained by adjunction $\mathrm{fib}_\Psi \dashv E_\Psi$ from the unique horn filler of f. Hence, $\Lambda(F)$ is a Grothendieck fibration.

The equivalence $(\Lambda(\Psi))_I \simeq \Lambda(\Psi_I)$ is obtained via identity on objects and via adjunction $\mathrm{fib}_\Psi \dashv E_\Psi$ on morphisms. ☐

Explicitly for coronal fibration $\mathcal{O}^{(n)} \xrightarrow{p} \mathcal{O}$, Proposition 5.4.8 states that the Leinster category construction Λ sends the below pullpack diagram on p (left) to the pullback diagram on $\Lambda(p)$ (right).

$$\begin{array}{ccc} (\mathcal{O}^{(n-1)})_I & \longrightarrow & \mathcal{O}^{(n)} \\ \downarrow & \ulcorner & \downarrow \\ \{I\} & \longrightarrow & \mathcal{O} \end{array} \quad \longmapsto \quad \begin{array}{ccc} (\Lambda(\mathcal{O}^{(n-1)}))_I & \longrightarrow & \Lambda(\mathcal{O}) \\ \downarrow & \ulcorner & \downarrow \\ \{I\} & \longrightarrow & \Lambda(\mathcal{O}) \end{array}$$

Construction 5.4.9. Suppose Φ and Ψ are perfect operator categories. Let us define the wreath product of Leinster categories $\wr_\Lambda$ such that $\Lambda(\Psi) \wr_\Lambda \Lambda(\Phi)$ consists of objects $(J, (A_j)_{j\in|J|})$ for $J \in \Phi$ and $B_i \in \Psi$. The morphisms of $\Lambda(\Psi) \wr_\Lambda \Lambda(\Phi)$ $(J, (A_j)_{j\in|J|}) \to (I, (B_i)_{i\in|I|})$ are tuples

$$(J \xrightarrow{\phi} I, (f_j : A_j \to B_{\phi(j)})_{j\in|J\times_{TI} I|})$$

for $\phi \in \Phi$ and $f_j \in \Psi$. Here the morphisms f_j are indexed only over points of $J \times_{TI} I$ the pullback of ϕ along the embedding $\iota_I : I \to TI$.

Proposition 5.4.10. *For perfect operator categories Φ and Ψ we have the following equivalence of categories:*

$$\Lambda(\Psi \wr \Phi) \simeq \Lambda(\Psi) \wr_\Lambda \Lambda(\Phi).$$

Proof. Let the functor $W : \Lambda(\Psi) \wr_\Lambda \Lambda(\Phi) \to \Lambda(\Psi \wr \Phi)$ be defined as follows:

- on objects W is the identity;
- on morphisms $(I \xrightarrow{\phi} J, (f_i : A_i \to B_{\phi(i)})_{i\in|I\times_{T_\Phi J} J|})$ for $\phi \in \Lambda(\Phi)$ and $f_i \in \Lambda(\Psi)$ the image is

$$(I \xrightarrow{\phi} J, (g_i : A_i \to N_{\phi(i)})),\ N_j = \begin{cases} T_\Psi M_j, & j \in |J| \subseteq |T_\Phi J|, \\ 1, & \text{otherwise}, \end{cases}$$

where $g_i = f_i$ for $i \in |I \times_{T_\Phi J} J|$ and $g_i : A_i \to 1$ is the unique map in Ψ for $i \notin |I \times_{T_\Phi J} J|$.

The functor W is bijective on morphisms and identity on objects, and hence is an equivalence of categories. □

There are other definitions of wreath product similar to that in 4.2.2. One such definition is the following.

Definition 5.4.11. The *categorical wreath product* is defined for the simplex category Δ and category A as seen in [2, Def. 3.1] and we shall denote it $\Delta \wr_c A$.

- The objects of $\Delta \wr_c A$ are tuples $([m], (a_1, ..., a_m))$ as for $[m] \in \Delta$ and $a_i \in A$.
- Morphisms are tuples $(\phi, (f_{ij} : a_i \to b_j))$ for $\phi \in \Delta$ and f_{ij} for $\phi(i-1) < j < \phi(i)$.

Definition 5.4.12. Joyal's Θ_n category is the iterated categorical wreath product $\Delta^{\wr n} := \Delta \wr_c ... \wr_c \Delta$ [2, Def. 3.3].

Proposition 5.4.13. *The Leinster category $\Lambda(\mathcal{O}^{(n)})$ is Θ_n^{op} [1, Ex. 6.8].*

Proof. Under the equivalence $\Delta \simeq \Lambda(\mathcal{O})^{\text{op}}$ in Proposition 5.4.3, Joyal's Θ_n category is equivalent to $\Lambda(\mathcal{O})^{(n)} := \Lambda(\mathcal{O}) \wr_\Lambda ... \wr_\Lambda \Lambda(\mathcal{O})$. Then the result follows from Proposition 5.4.10. □

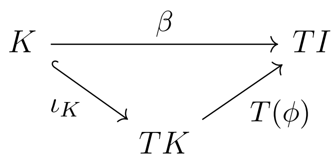
This is a beautiful result we have been building to for a few sections now. This is a whole family of objects controls space of $\mathcal{O}^{(n)}$-monoidal structures, from non-symmetric monoidal objects to symmetric monoidal objects. This should remind the reader of the class of operads $\mathbb{E}_n$ we began with; the next chapter will give the connection between them in more detail.

5.5 Quasioperads and their Algebras

Definition 5.5.1. Suppose Φ is a perfect operator category, let $\beta_T : J \to K \in \Lambda(\Phi)$ be the morphism given by $\beta : J \to TK \in \Phi$. We call β_T *inert* if $J \times_{TK} K \to K$ is an isomorphism in the following pullback square [1, Def. 7.1].

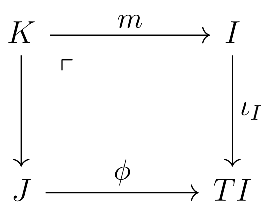
$$\begin{array}{ccc} J \times_{TK} K & \longrightarrow & K \\ \downarrow & & \downarrow{\scriptstyle \iota_K} \\ J & \xrightarrow[\beta]{} & TK \end{array}$$

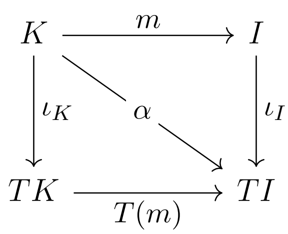
Let $\alpha_T : K \to I \in \Lambda\Phi$ be the morphism $\alpha : K \to TI \in \Phi$ then α_T is called *active* if there exists a map $\phi : K \to I \in \Phi$ such that the following diagram commutes [1, Def. 7.1].



Proposition 5.5.2. *For all $J \xrightarrow{\phi_T} I \in \Phi$ there exists a unique inert-active factorisation up to unique isomorphism [1, Lem. 7.3].*

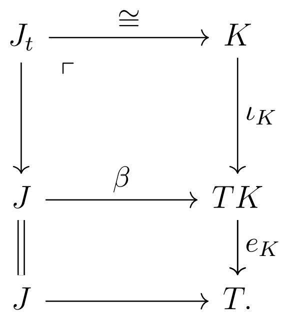
Proof. (See proof of [1, Lem. 7.3].) Let $K := J \times_{TI} I$ in the pullback square below (left) and consider the composite $\alpha := \iota_I \circ \phi$ in the adjacent diagram.





Then $\alpha_T \in \Lambda(\Phi)$ gives an active morphism $\alpha \in \Lambda(\Phi)$.

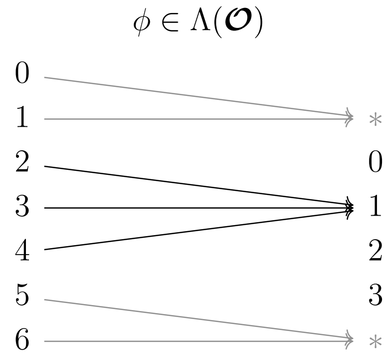
Now choose a morphism $J \to T$ such that $J_t \cong K$ then we have that the outer and upper squares are pullback squares in the following commutative diagram.



Then the morphism β corresponding to $J_t \cong K$ via the adjunction $U \dashv E$ where $T_\Phi = U \circ E$, gives us our inert-active factorisation, $J \xrightarrow{\phi_T} I = J \xrightarrow{\beta_T} K \xrightarrow{\alpha_T} I$. $\square$

Example 5.5.3. Consider the morphism ϕ in the diagram below.

Simply put, the inert morphism corresponds to a restriction and the active morphism maps only onto points which have not been added by T.



Then the inert-active factorisation $\phi_T = \alpha \circ \beta$ is shown in Figure 5.5.3.

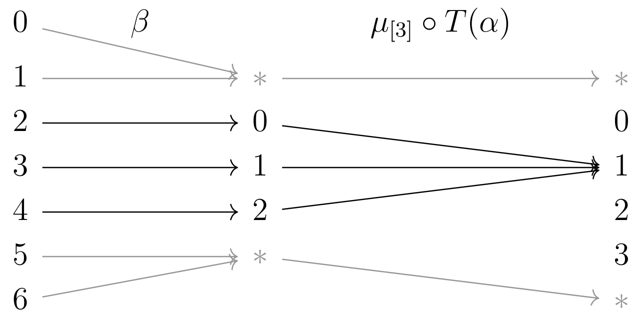


Figure 5.2: Inert-active factorisation of ϕ: inert map β and active map α.

Notation 5.5.4. With respect to a functor $q : X \to S$ let $\mathrm{Hom}_X^{\phi}(x, y)$ denote the union of connected components that cover the connected component of $\phi \in S$.

Definition 5.5.5. A Φ–quasioperad or ∞–operad over Φ is an inner fibration $X^{\otimes} \xrightarrow{p} N\Lambda(\Phi)$ satisfying the following conditions [1, Def. 7.8]:

(1) For every inert morphism $\phi : J \to I$ in $\Lambda(\Phi)$ and for every object $x \in X_J^{\otimes}$ there is a p-cocartesian edge $x \to y$ covering ϕ.

(2) For any objects $I, J \in \Phi$, any objects $x \in X_I^{\otimes}$ and $y \in X_J^{\otimes}$, any morphism $\phi : J \to I$ of $\Lambda(\Phi)$, and any p–cocartesian edges $\{y \to y_i | i \in |I|\}$ lying over the inert morphisms $\{p_i : I \to \{i\} | i \in |I|\}$, the induced map

$$\mathrm{Hom}_{X^{\otimes}}^{\phi}(x, y) \to \prod_{i \in |I|} \mathrm{Hom}_{X^{\otimes}}^{p_i \circ \phi}(x, y_i)$$

is an equivalence.

(3) For any object $I \in \Phi$, the p–cocartesian morphisms lying over the inert morphisms $\{I \to \{i\} | i \in |I|\}$ together induce an equivalence

$$X_I^{\otimes} \to \prod_{i \in |I|} X_{\{i\}}^{\otimes}.$$

We will call functor $X^{\otimes} \xrightarrow{p} \Lambda(\Phi)$ an *ordinary* Φ*-operad* if there exists a p-cocartesian lift of each inert morphism in $\Lambda(\Phi)$ and p satisfies the Segal condition. Where the distinction is not important we shall refer to the collection of ordinary Φ-operads and Φ-quasioperads just as Φ-operads.

Notation 5.5.6. Let us denote the image of $[n]$ under the equivalence $\Delta \to \Lambda(\boldsymbol{\mathcal{O}})^{\text{op}}$ as $[n]^{\circ} := \{1 \leq 2 \leq ... \leq n\}$.

Remark 5.5.7. Conditions (2) and (3) give us the Segal condition. More explicitly, condition (3) says that for every $[n] \in \Delta^{\text{op}}$ objects in $X_{[n]}^{\otimes}$ are of the form $(a_i)_{i \in [n]^{\circ}}$ for $a_i \in X_1$.

Definition 5.5.5 gives us a fibrational definition of a Φ-operad, but in Chapter 1 we introduce operads as multicategories. So how do we relate the two?

Example 5.5.8. We shall specify the structure and verify the properties of the multicategory determined by quasioperad $X^{\otimes} \xrightarrow{p} \Lambda(\boldsymbol{\mathcal{O}})$, call this multicategory $\boldsymbol{\mathcal{X}}$.

Let elements in the fibre X_1 be objects in the underlying category of $\boldsymbol{\mathcal{X}}$. We define the multihom sets as follows: $\boldsymbol{\mathcal{X}}(a = (a_i)_{i \in I^{\circ}}, b) := \text{Hom}_{X^{\otimes}}(a, b)$ for $a \in X_I^{\otimes}$ and $b \in X$. Condition (2) says that we may interpret a morphism in $\Lambda(\boldsymbol{\mathcal{O}})$ as a collection of multimorphisms as in Figure 5.5.8.

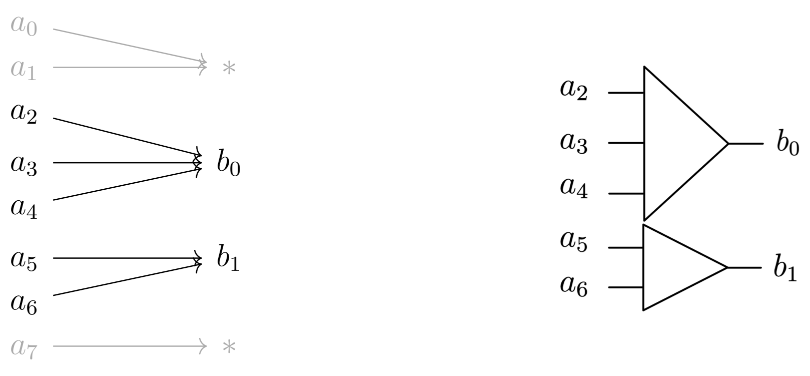
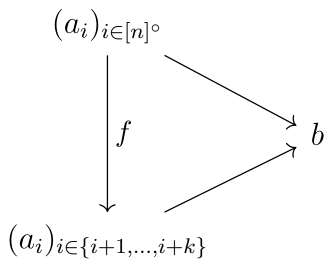


Figure 5.3: Equivalence $\text{Hom}^{\phi}(a, b) \to \text{Hom}^{p_0 \circ \phi}(a, b_0) \times \text{Hom}^{p_1 \circ \phi}(a, b_1)$ for $a = (a_i)_{i \in [7]}$, $b = (b_i)_{i \in [1]}$

As p is an inner fibration then we can lift composition in $\Lambda(\mathcal{O})$ to composition in $X^{\otimes}$. The cocartesian lifts of inert morphisms $\phi : J \to I$ give us commuting triangles



which allow us to obtain k-fold morphisms from n-fold morphisms for $k \leq n$ by restricting the domain.

5.5.9. Endowing objects with a quasicategory structure equips them with a notion of homotopy; much of the success of the construction of Barwick's theory of operads lives in the homotopy theory it admits. One of the key results of Barwick's paper sheds some light on how this theory of operads allows us to organise $\mathbb{E}_n$-structures.

Theorem 5.5.10. The homotopy theory of $\mathcal{O}^{(n)}$-algebras is equivalent to that of $\mathbb{E}_n$-algebras.

Proof. See [1, Ex. 8.13] for the statement and proof. □

Remark 5.5.11. This result builds on a model of Φ-operads introduced in [1, §2] for operator categories Φ (not necessarily perfect).

There is a statement which alludes to this relationship, but which is slightly easier to understand heuristically.

Proposition 5.5.12. *The following are canonically equivalent:*

- *algebras over terminal $\mathcal{O}^{(n)}$-quasioperad;*
- *algebras over the operad $\mathbb{E}_n$.*

Proof. See [1, Ex. 2.16] for the precise statement and proof. □

There are a few more families of objects which are equivalent to those in Proposition 5.5.12 above that will give us motivation for possible future research directions, as explained in Chapter 5.6.

5.6 Φ-monoidal categories vs Φ-operads

5.6.1. In this section we will generalise our example of an $\boldsymbol{\mathcal{O}}$-monoid in Section 2.3 to Φ-monoidal categories for perfect operator categories Φ. We also make explicit the role of the inert and active morphisms in our inherent factorisation system in the Leinster category $\Lambda(\Phi)$ in differentiating between Φ-operads and Φ-monoidal categories.

Definition 5.6.2. For perfect operator category Φ, an ordinary Φ-*monoidal category* is a cocartesian fibration $X^{\otimes} \xrightarrow{p} \Lambda(\Phi)$ satisfying the Segal condition. A Φ-*monoidal* ∞-*category* is a cocartesian fibration $X^{\otimes} \xrightarrow{p} N\Lambda(\Phi)$ satisfying the Segal condition. (See an equivalent construction in [1, Not. 8.11]. This is a generalisation of the construction seen in [17, p. 166-167].)

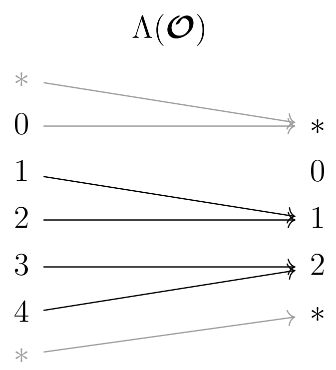
Remark 5.6.3. We these definitions separately though they are very similar to highlight the fundamental role of the nerve functor in their difference.

Remark 5.6.4. Thus we get a generalisation of the definition of a symmetric monoidal category as in [17, Def. 2.0.0.7].

Remark 5.6.5. The only difference in the definitions of Φ-(quasi)operads and Φ-monoidal (∞-) categories, is the cocartesian lifts of active morphisms.

Example 5.6.6. Let us return to an example of an $\boldsymbol{\mathcal{O}}$-monoidal category that began our story: $\mathbf{Vect}^{\otimes}$. The morphism $\mathbf{Vect}^{\otimes} \to \Delta^{\mathrm{op}}$ in 2.2.2 is, in fact, an ordinary cocartesian fibration [.] In particular, all the inert maps have a cocartesian lift. This fibration also satisfies the Segal condition for ordinary categories and so $\mathbf{Vect}^{\otimes}$ is an ordinary one object $\boldsymbol{\mathcal{O}}$-operad.

Observation 5.6.7. It is in some ways more natural to consider $\mathbf{Vect}^{\otimes} \to \Lambda(\boldsymbol{\mathcal{O}})$ rather than over Δ^{op} as the morphisms in $\mathbf{Vect}^{\otimes}$ are defined more directly in terms of maps in $\Lambda(\boldsymbol{\mathcal{O}})$ as in the diagram below.



For example the cocartesian lift at $(V_i)_{i\in[4]}$ of the morphism in 5.6.7 above is just $(V_i)_{i\in[4]} \to (\Bbbk, V_1 \otimes V_2, V_3 \otimes V_4)$ where one takes the tensor of vector spaces in the fibre over each point in [2].

5.6.8. We will now detail the importance in the difference between cocartesian lifts of inert and active morphisms. In our specification of a Φ-operad as a multicategory, the cocartesian lifts of inert morphisms correspond to restrictions on multimorphisms to a subset of domain objects.

Inert Morphisms. Let $f : x \to y$ be an arbitrary map lying over $[n] \to [m]$. Then $[n] \to [m]$ factors uniquely as an inert $i : [n] \to [k]$ following by an active $a : [k] \to [m]$. Then since we can lift i to a cocartesian edge i_*, we can also lift a to a_* such that $f = a_* i_*$. In an arbitrary morphism in $X^{\otimes}$ not all objects correspond to the inputs of multimorphisms. Being able to lift the factorisation of morphisms allows us to associate each morphism with an active morphism. In a lift over an active morphism all objects in the domain correspond to inputs of multimorphisms.

Active and Inert Morphisms. In order to define a ϕ-monoidal category we need the cocartesian lifts of active morphisms to construct functors between fibres that give us the compatibility data of the product, e.g. associativity and unitality for $\Phi = \mathcal{O}$. By comparison, we consider commuting functor diagrams which give us different ways to restrict the domain of morphisms of X. Different inert factorisations of a morphism in $N\Phi$ give different but equivalent restrictions of morphism domains (see Figure 5.6.

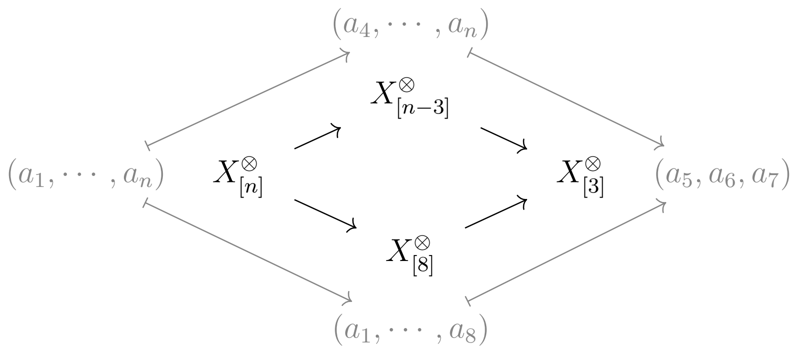


Figure 5.4: Equivalent morphisms satisfying the commutative diagram of restrictions as above.

5.6.9. To summarise, though the interpretations of the fibrations that define operads and monoidal categories feel different they exploit properties of the same factorisation system. It makes sense that operads require less structure from the offset because, as we saw in Chapter 1 on operads, specifying an algebra over an operad requires introducing more structure. That structure coincides with specific lifts of the active morphisms in our underlying Leinster category.

Future Research Directions

It was anticipated that other operator categories would give rise to a whole array of useful operads that current research cares about, such as cyclic operads [8] or $\mathcal{LM}^{\otimes}$ operads [17, §4.2.1]. However, it is expected that, in the case of cyclic operads, the operator category formulation as it stands is not flexible enough to account for equivalence of operations under cyclic permutation of inputs with outputs. As for $\mathcal{LM}^{\otimes}$ operads, well, it has simply not been thought about enough yet in this context.

As we have already seen, $\mathbb{E}_k$-algebras were invented to detect loops. The May Recognition Theorem is roughly that grouplike $\mathbb{E}_n$-algebras are exactly the same thing as n-fold loop spaces $\Omega^n X$ [21]. This is the relationship we briefly mentioned in the motivation for operads in this project. However, from correspondence with Barwick it is clear that there is more to the story which has not yet been explored, as we now explain.

The following are 'the same':

- algebras over the operad E_n;
- algebras over the terminal $O^{(n)}$-operad;
- n-tuply monoidal ∞-groupoids;
- (∞, n)-categories with one object, one 1-morphism, ..., and one $n-1$-morphism;
- locally constant factorization algebras on $\mathbb{R}^n$.

In this project we saw a glimpse of some of these relationships and it may be read about in more detail in [16]. However, even in the most current research some are not given explicitly or directly. For example, it is unclear how locally constant factorisation algebras are connected to the combinatorics of the operator category $\boldsymbol{\mathcal{O}}^{(n)}$. A new mathematical framework around objects called *worlds* is currently being developed by Barwick. This new framework may also have the scope to address a type of duality called Koszul duality between the Lie operad, which gives us lie algebras, and the $\mathbb{E}_\infty$ (terminal symmetric) operad.

Conclusion

Our objective in this project was to give an exposition on some key results conceptual cornerstones in the theory of operads in Barwick's 'From operator categories to higher operads'. Specifically, we explicated the elegance with which the theory ties together existing theories of operads and the subtlety in defining types of algebraic objects for a given operator category.

- Elegance: the tiny amount of combinatorial data of $\mathcal{F}$ recovers May's existing theory of symmetric operads [21]; $\mathcal{O}$ recovers non-symmetric operads [21]; the generalisation and much more natural reformulation of the wreath product then allows us to recover $\mathcal{O}^{(n)}$ and extends the theory of $\mathbb{E}_k$-operads.
- Subtlety: we saw how cocartesian lifts of inert morphisms are enough in the definition of a Φ-operad, but that cocartesian lifts of active maps are also required to construct a Φ-monoidal category.

Finally, we look forward to seeing this theory of operator categories extended as we search for an explanation of their relationship to factorisation algebras and Koszul duality.

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