Homotopy theory and higher categories: The combinatorial elegance of the simplex category.

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Introduction

The world of homotopy coherent category theory can be modelled in many different ways. One of the most powerful involves the use of the category of simplices and simplicial sets which allow us to combinatorially encode and express homotopical properties.

We begin by detailing simplicial homotopy, and then go on to look into the world of model categories.

Our main model category of interest will be the Quillen model category of simplicial sets which as a model category is equivalent to the category of topological spaces. We will then introduce the complete n-fold Segal space model for (∞, n)-categories using this model category of simplicial sets.

0.1 Simplicial sets and spaces

1. For the same reasons that we enjoy working with CW complexes instead of general topological spaces we might want to abstract further and consider a combinatorial description of homotopy. In turns out that simplicial sets are exactly the right tool for this purpose. In this first section we will build towards the result that the homotopy theory of simplicial sets is 'equivalent' to the homotopy theory of spaces.

The results and definitions in this first section may be found in [2, Goerss-Jardine].

Definition 2. The simplex category Δ is the category whose objects are linearly ordered sets $[n] := \{0 < 1 < ... < n\}$ and whose morphisms are order preserving functions.

Definition 3. A *simplicial set* is a functor $X : \Delta^{op} \to \mathbf{Set}$. These form the category **sSet** where the morphisms are natural transformations, $(X([n]) \to Y([n]) \in \mathbf{Set})_{n \in \mathbb{N}}$.

Definition 4. More generally, for any category $\mathcal{C}$ a *simplicial object* is a functor $\Delta^{op} \to \mathcal{C}$.

Example 5. One fundamental example of a simplicial set is $\Delta^n : \Delta^{op} \to \mathbf{Set}$ where $\Delta^n([k])$ is the set of $\binom{n}{k}$ elements which correspond to the k-dim faces of the convex hull of $n + 1$ points.

Construction 6. The above example hints at the fact that the simplex category Δ embeds into **sSet** by the Yoneda embedding よ$: \Delta \to \mathbf{sSet}; [n] \to \mathbf{Map}(-, [n])$ where the maps $[n] \to [m]$ correspond 1-1 with maps $\Delta^n \to \Delta^m$.

Example 7. Another example is the singular simplicial set of a space. For a space X we define $\mathrm{Sing}(X) : \Delta^{op} \to \mathbf{Set}$ sends $[n]$ to $\mathrm{Map}(|\Delta^n|, X) \cong \mathrm{Map}(\mathrm{D}^n, X)$.

Construction 8. Define the *geometric realisation* functor $|\cdot| : \Delta \to \mathbf{Top}$ as following:

- the set $[n] \in \Delta$ is sent to the subspace $\{(x_0, ..., x_n) | \sum_i x_i = 1, x_i \geq 0\} \subseteq \mathbb{R}^{n+1}$ with the subspace topology;
- the morphism $\phi : [n] \to [m]$ is sent to $|\phi|; (t_0, ..., t_n) \mapsto (s_0, ..., s_m)$ where

$$s_i = \begin{cases} 0, & \phi^{-1}(i) = \varnothing, \\ \sum_{\phi^{-1}(i)}, & \text{otherwise.} \end{cases}$$

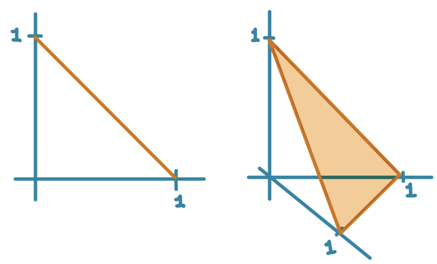


Figure 1: Geometric realisation of Δ^1 and Δ^2.

The following proposition formalises the intuition given in Figure 2 that simplicial sets arise from the gluing of simplices.

Proposition 9. *Every simplicial set can be written as a colimit of simplicial sets in* よ*(*Δ*).*

10. Analogous to constructing a CW complex (using pushouts of copies of D^n and attaching via boundary maps) we may also 'glue' together simplicial sets using Δ^n's. The key differences are the following:

- simplicial sets are not spaces, they are combinatorial abstractions of CW complexes;
- unlike CW complexes where the gluing maps can be any maps from the boundary of a cell, the gluing in of simplicial sets is specified by mappings on vertices, which are linearly interpolated when you take the geometric realisation.

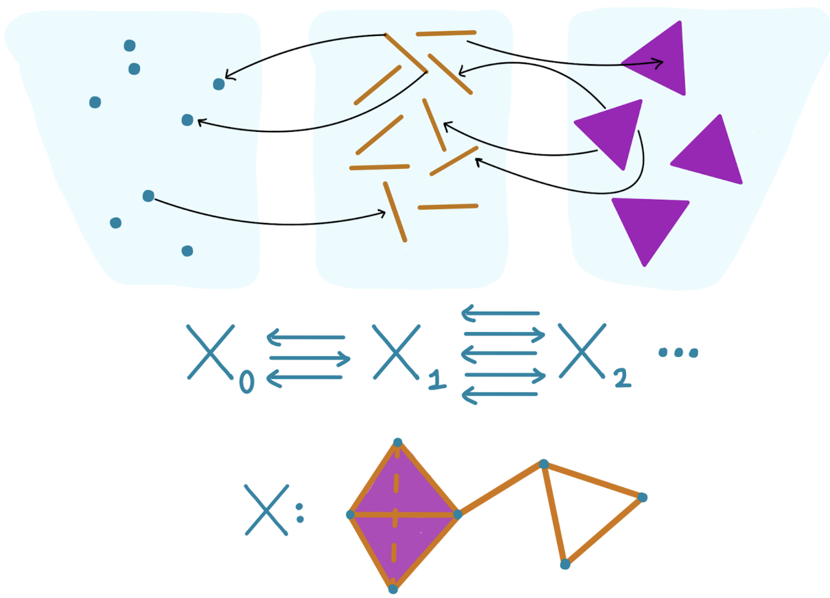


Figure 2: Simplicial set intuition: 0-cells, 1-cells, 2-cells.

For a simplicial set X, $X_m := X([m])$ is the set of m-cells and the image of inclusion $[m] \hookrightarrow [m+1] \in \Delta$ under X $(X_{m+1} \to X_m)$ give the attaching maps.

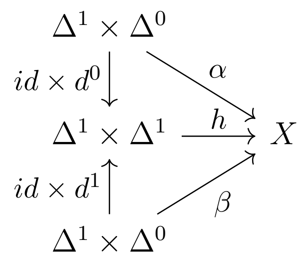
Definition 11. A *kan fibration* is a morphism of simplicial sets $X \xrightarrow{p} Y$ that satisfies the right lifting property with respect to all $\wedge_k^n$ horn inclusions.

$$\begin{array}{ccc} \wedge_k^n & \longrightarrow & X \\ \big\downarrow & \overset{\exists}{\nearrow} & \big\downarrow p \\ \Delta^n & \longrightarrow & Y \end{array}$$

Definition 12. The functor Sing : **Top** $\to$ **sSet** sends a topological space X to the its singular simplicial set $\text{Sing}(X) : \Delta^{op} \to \textbf{Set}; [n] \mapsto \text{Map}(|\Delta^n|, X)$.

Definition 13. For paths $\alpha, \beta : \Delta^1 \to X$ in Kan complex X, a *(simplicial) homotopy*

$h : f \simeq g$ is the data of the following commutative diagram.



Here d^0 and d^1 are specified by inclusions $[0] \hookrightarrow [1]$ into 0 and 1 respectively.

Kan complexes are spaces

Remark 14. The condition for having all $\wedge_k^n$ horn fillers specifies that you can concatenate $(1 \leq k \leq n-1)$ and invert $(k \in \{0, n\})$ all paths and homotopies up to homotopy.

Lemma 15. *For all Kan complexes K we have an isomorphism of homotopy groups induced by geometric realisation:*

$$\pi_n(K, v) \cong \pi_n(|K|, v).$$

Proof. By induction.
Base case:
Injectivity: A simplicial set X may be expressed as the coproduct of its connected components $\amalg_i X^i$. Geometric realisation is a left adjoint and so preserves colimits, in particular it preserves coproducts and so $| \amalg_i X^i| \cong \amalg_i |X^i|$. Therefore, if $|X|$ has only one connected component, X does too, and so $\pi_0(|.|)$ is injective.

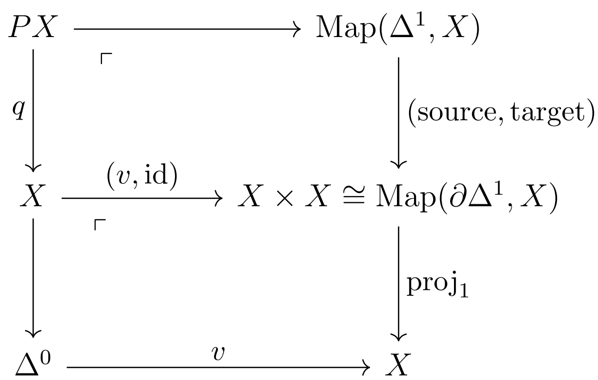
Surjectivity: Any point in $y \in |X|$ must belong to the geometric realisation of some simplex $\Delta^n \xrightarrow{\alpha} X$. Since $|\Delta^n|$ is path-connected for all n, the point y belongs to the same path component of some vertex $|v| \in \alpha$. Hence, v is in the preimage of y, and so $\pi_0(|.|)$ is surjective.

Inductive step:
For Kan complex K consider the path space fibration $PK \xrightarrow{q} K$:

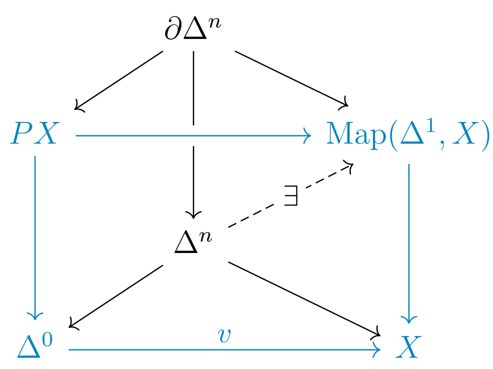
- we define $PK := \{[\gamma] | \gamma \text{ is a path in } K \text{ starting at } v \in K_0\}$;
- the notation $[.]$ means taking the rel $\partial\Delta^1$ homotopy class;
- and q is projection onto the endpoint of γ.

(We saw the analogous construction in Hatcher to show existence of a universal covering space in the proof of the Galois correspondence [3, p.64].)

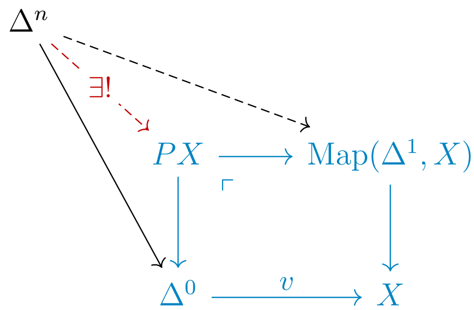
Now we show that $\pi_n(PK) = 0$ for all $n \in \mathbb{N}$. The following is a commutative diagram of pullbacks, where q is our path space fibration.



Now we use that the vertical map $\mathrm{Map}(\Delta^1, X) \xrightarrow{\text{target}} X$ has the right lifting property with respect to all boundary inclusions $\partial\Delta^n \to \Delta^n$. This implies that $PX \to X \to \Delta^0$ has the same property. We have that the following diagram commutes.



Since the front face (above, in blue) is a pullback square we get a unique map (below, in red) that makes the whole diagram commute.



Then, considering the fibre over v, ΩK, we use the long exact sequence of homotopy groups from the fibration $\Omega K \hookrightarrow PK \xrightarrow{q} K$ [3, Thm.4.41] to get isomorphism

$$\underset{\pi_{n+1}(PK, v)}{0} \longrightarrow \pi_{n+1}(K, v) \cong \pi_n(\Omega K, v) \longrightarrow \underset{\pi_n(PK, v)}{0}$$

We may do exactly the same for fibration $|\Omega K| \hookrightarrow |PK| \xrightarrow{q} |K|$ to get

$$\pi_{n+1}(|K|, v) \cong \pi_n(|\Omega K|, v).$$

Assuming, $\pi_n(K, v) \cong \pi_n(|K|, v)$ for all K we conclude that

$$\pi_{n+1}(K, v) \cong \pi_n(\Omega K, v) \cong \pi_n(|\Omega K|, v) \cong \pi_{n+1}(|K|, v).$$

Thus our proof by induction is complete.

□

Proposition 16. *The counit of the adjunction $|Sing(X)| \to X$ is a weak equivalence of spaces.*

Proof. We use the adjunction $\mathrm{Hom}_{\mathbf{sSet}}(X, Sing(Y)) \cong \mathrm{Hom}_{Top}(|X|, Y)$ on maps $\Delta^n \to \mathrm{Sing}(X)$ to get the isomorphism $\pi_n(Sing(X), v) \to \pi_n(X, v)$ and using Lemma 15 take the composite

$$\pi_n(|Sing(X)|, v) \cong \pi_n(Sing(X), v) \xrightarrow{\cong} \pi_n(X, v).$$

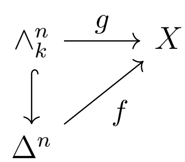
□

Corollary 17. Every space X is weakly equivalent to the geometric realisation of a Kan complex.

18. There are equivalent simplicial formulations of many major theorems in homotopy theory, such as Van Kampens theorem. However, in the simplicial setting you have access to powerful combinatorial machinery that you cannot access in the usual homotopy theory of spaces. Even with these tools, since the homotopy theories of both are equivalent, we may utilise all the homotopical results of simplicial homotopy theory for spaces.

19. Another way to look at categories which admit a notion of homotopy is to consider higher categories, and not just categories with 1-morphisms, 2-morphisms, or even 3-moprhisms, but categories with k-morphisms for every $k \in \mathbb{N}$.

Definition 20. Let X be a simplicial set.



If the diagram above commutes then f is called a *horn filler*. If for every map $\wedge^n_k \to X$ there exists an extension $\Delta^n \to X$ then X satisfies the $\wedge^n_k$ *horn filler condition* (often called the *Kan extension condition* [5, p. x, 8]).

Definition 21. If simplicial set X satisfies the $\wedge^n_k$ horn filler condition for all $n \geq 2$ and $n \geq k \geq 0$ then X is an ∞-*groupoid*. This definition coincides with the *Kan complex* model for ∞-groupoids [5, Def. 1.1.2.1].

Definition 22. For simplicial set X, if the $\wedge^n_k$ horn filler condition is satisfied for $0 < k < n$, $n \geq 2$ then X is an $(\infty, 1)$-*category* [5, Def. 1.1.2.4]. Here we have given the *quasicategories* model of $(\infty, 1)$-categories.

23. We will now consider a different invariants of spaces which are a generalisation of ordinary homology. We will see which types of homotopy are important here.

0.2 Factorisation homology

24. Ordinary or *generalised* homology or homotopy may be a good invariant for topological spaces in general, but when dealing with manifolds we would like our invariants to respect diffeomorphism but not homotopy equivalence, to respect isotopy but not homotopy. None of the invariants we have seen so far are sensitive enough for such a task so let us define another.

The definitions and results of this section may be found in [7, Tanaka].

Definition 25. The symmetric monoidal category $Disk_{n,fr}$ has objects disjoint unions of $\mathbb{R}^n$ and morphisms embeddings where the fr denotes we may equivalently consider our objects as n-cubes I^n and our embeddings as rectilinear embeddings as in Figure 3.

In particular, $Disk_{n,fr}$ contains the 0-fold disjoint union of R^n, the empty manifold $\varnothing$. The monoidal product is disjoint union $\amalg$ with $\varnothing$ as the unit.

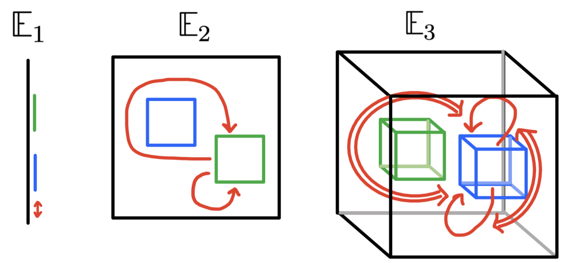


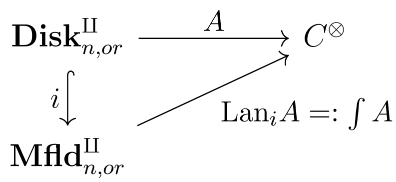
Figure 3: Rectilinear embeddings and isotopies of two n-cubes into an n-cube for $n = 1, 2, 3$.

Proposition 26. *A symmetric monoidal functor from $Disk_{n,fr}$ to any other symmetric monoidal category $\mathcal{C}^{\otimes}$ is an $\mathbb{E}_n$-algebra in $\mathcal{C}^{\otimes}$.*

Example 27. Take $\mathbf{Vect}_{\Bbbk}^{\otimes}$ the category of vector spaces over field $\Bbbk$ and linear transformations with the tensor product. Then a symmetric monoidal functor $F : Disk_{n,fr}^{\amalg} \to \mathbf{Vect}_{\Bbbk}^{\otimes}$ is a $\Bbbk$-algebra where the embedding $\mathbb{R} \amalg \mathbb{R} \to \mathbb{R}$ gives the multiplication $A \otimes A \to A$ under F.

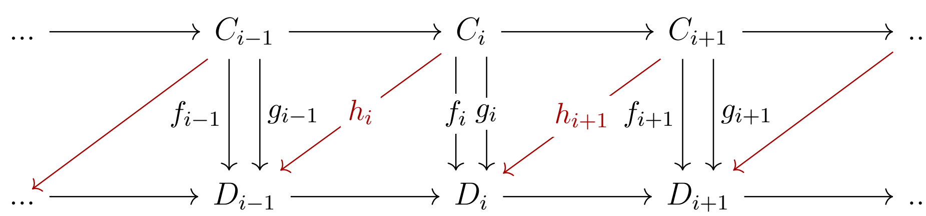
For $n = 1$ A is non-commutative and for $n \geq 2$ A is commutative. In all cases, multiplication, associativity, and unitality are up to isomorphism of vector spaces.

Definition 28. Let A be a $\mathbb{E}_n$-algebra $A : Disk_{n,fr}^{\amalg} \to C^{\otimes}$. The *$n$-dimensional factorisation homology* with coefficients in A of manifold X is the left Kan extension of A along inclusion $Disk_{n,fr}^{\amalg} \hookrightarrow Mlfd_{n,fr}^{\amalg}$.



Example 29. Vect is an easy example to give but it loses some of the higher homotopical information of our spaces and embeddings. Lets give the example of ∞-category Chain_k instead.

Objects of $\mathrm{Chain}_{\Bbbk}$ are cochain complexes $(C_\bullet, \partial_C)$ of vector spaces over k, 1-morphisms are chain maps $(f_i : C_i \to D_i)_{i\in\mathbb{N}}$, 2-morphisms are chain map homotopies $(h_i : C_i \to D_{i-1})_{i\in mathbbN}$ such that $\partial_D h + h\partial_C = f - g$, and k-morphisms for $k \geq 1$ are degree $-k$ maps from $C_\bullet \to D_\bullet$ for which the same equation holds where f and g are $k-1$-morphisms.



Remark 30. For a functor $F : Disk_{n,fr} \to \mathbf{Chain}^{\otimes}_{\Bbbk}$ embeddings in $\mathrm{Disk}_{n,fr}$ are sent to chain maps, isotopies of embeddings are sent to chain map homotopies, and so on.

Remark 31. We have that $Disk_{n,fr}$-algebras are precisely $\mathbb{E}_n$-algebras. Preserving framing is analogous to rectilinearity of embeddings in the little n-cubes operad.

Lemma 32. *Assume $C^{\otimes}$ is an ∞-category admitting all sifted colimits and $\otimes$ commutes with sifted colimits in each variable. Then factorization homology can be made symmetric monoidal; we may supply equivalences:*

$$\int_{X \amalg Y} A \simeq \int_X A \otimes \int_Y A.$$

Theorem 33. Excision. Given the symmetric monoidal left Kan extension of such a $C^{\otimes}$ then the factorisation homology $\int A$ is a local-global invariant in the following way:

$$\int_{X \amalg_{R\times W} Y} A \simeq \int_X A \otimes_{\int_{R\times W} A} \int_Y A.$$

It preserves certain nice pushouts in **Mfld**, where for an $n-1$-manifold W the intersection of manifolds is homeomorphic to $W \times \mathbb{R}$. For $C^{\otimes} = \mathbf{Vect}^{\otimes}$ these pushouts are the relative tensor product of $\mathbb{k}$-algebras.

Remark 34. We also may conclude that for $\mathbf{Vect}^{\otimes}$ the factorisation homology such that $\int_{\mathbb{R}^n} = A$ on $\mathbb{R}^n$ with the reverse orientation is A^* the dual of A.

Example 35. Let us compute the factorisation homology of the circle with coefficients in $A \in \mathbf{Vect}_{\mathbb{k}}^{\otimes}$.

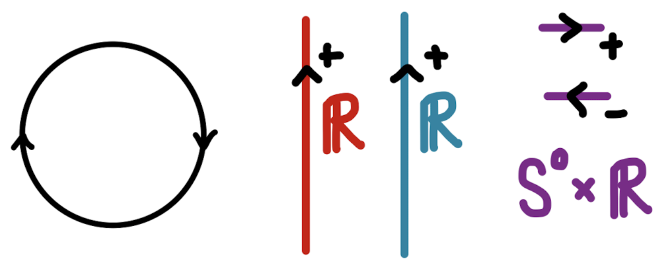


Figure 4: Oriented circle, two positively oriented copies of $\mathbb{R}$, $S^0 \times \mathbb{R}$ where S^0 has the standard orientation $\{+, -\}$.

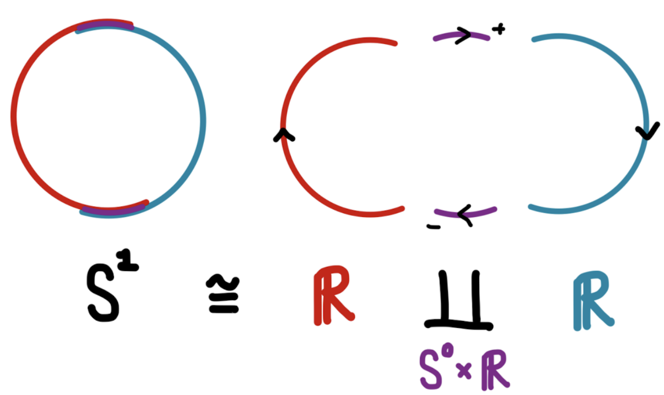


Figure 5: Decomposition of the circle in $\mathbf{Mfld}^{\amalg}_{n,or}$.

Preserving framing is the same as preserving orientation in this example. So copies of $\mathbb{R}_-$ (with negative orientation) must be embedded into positively oriented copies of $\mathbb{R}^+$ in the reverse direction, as in the figure below. We have that

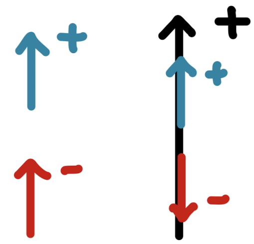


Figure 6: Oreintation preserving embedding $\mathbb{R}_+ \amalg \mathbb{R}_- \to \mathbb{R}^+$.

$$\int_{S^1} A \simeq \int_{\mathbb{R}\amalg_{S^0\times\mathbb{R}}\mathbb{R}} A \simeq \int_{\mathbb{R}} \bigotimes_{S^0\times\mathbb{R}} \int_{\mathbb{R}} A \simeq A \bigotimes_{A\otimes A^*} A.$$

36. We may in general consider factorisation homology on oriented manifolds or in fact on any class of manifolds whose tangent spaces admit a G-action for a topological group G.

Definition 37. Suppose we have a map of topological groups $G \xrightarrow{p} Gl_n(R)$ then a **tangential G-structure** on a smooth manifold X is the data of a commutative triangle and homotopy:

$$\begin{array}{ccc} BG & \xrightarrow{B(p)} & BGl_n(\mathbb{R}) \\ {\scriptstyle \phi_X}\uparrow & \simeq \nearrow & \\ X & & \end{array}$$

Remark 38. For $G = SO(n)$ we consider manifolds equipped with an orientation $\mathbf{Mfld}_{n,or}$. For $G = *$ we consider manifolds equipped with a framing $\mathbf{Mfld}_{n,fr}$.

Remark 39. A map which factors through $BG = \{*\}$ is a trivialisation of X. Therefore, manifolds which admit a framing have tangent bundles which are essentially constant, meaning that we have a homotopy between a trivialisation of X and its tangent bundle.

Remark 40. The classifying space of $Gl_n(\mathbb{R})$, $BGl_n(\mathbb{R})$ is $Gr_n(\mathbb{R}^\infty)$, the space of subspaces of $\mathbb{R}^n$. The classifying space of $SO(n)$ is $\widetilde{Gr_n}(\mathbb{R}^\infty)$: the space of subspaces of $\mathbb{R}^n$ with positive orientation, by which we mean that the matrix of basis vectors of the subspace has positive determinant.

So an orientation on a manifold is such that the tangent spaces at each point all have positive orientation, or all have negative orientation.

Lemma 41. *We have the following homotoopy equivalence of orientation preserving embeddings of $\mathbb{R}^n$ into itself, with the special orthogonal topological group:*

$$Emb^{or}(\mathbb{R}^n, \mathbb{R}^n) \simeq SO(n).$$

Proof. Let $Emb_0^{or}(\mathbb{R}^n, \mathbb{R}^n)$ denote the embeddings which fix the origin then there exists a homotopy $h(j,t) = j - tj(0)$ which gives the equivalence

$$Emb^{or}(\mathbb{R}^n, \mathbb{R}^n) \simeq Emb_0^{or}(\mathbb{R}^n, \mathbb{R}^n).$$

Then take the deformation retraction $h'(j,t)(v) = \frac{j(0+tv)-f(0)}{t} = \frac{f(tv)}{t}$ which gives us the equivalence

$$Emb_0^{or}(\mathbb{R}^n, \mathbb{R}^n) \simeq Gl_n^+(\mathbb{R}).$$

The final equivalence

$$Gl_n^+(\mathbb{R}) \simeq SO(n)$$

is given by the Gram-Schmidt process. □

Lemma 42. *The space of frame preserving embeddings of $\mathbb{R}^n$ into itself $Emb^{fr}(\mathbb{R}^n, \mathbb{R}^n)$ is contractible.*

Proof. Similar to the proof above for oriented embeddings we just notice that preserving framing implies there is no rotation, so every embedding is in the homotopy class $\mathbb{1} \in SO(n)$. $Emb^{fr}(\mathbb{R}^n, \mathbb{R}^n) \simeq Emb_0^{fr}(\mathbb{R}^n, \mathbb{R}^n) \simeq Gl_n^+(\mathbb{R}) \simeq *$. □

Remark 43. For the framed case, by the data of the homotopy of the framing there is always a homotopy to an embedding which preserves the trivialisation of $\mathbb{R}^n$.

Remark 44. It seems that having a tangential G-structure ensures that any G-structure compatible decomposition of a manifold gives us the same invariant in our factorisation homology.

So let us give the most general definition with our new understanding of tangential G-structures.

Definition 45. Let A be a $Disk_{n,G}$-algebra $A : Disk^{\amalg}_{n,G} \to C^{\otimes}$. The *$n$-dimensional factorisation homology* with coefficients in A of manifold X is the left Kan extension of A along inclusion $Disk^{\amalg}_{n,G} \hookrightarrow Mlfd^{\amalg}_{n,G}$.

$$\begin{array}{ccc} \mathbf{Disk}^{\amalg}_{n,or} & \xrightarrow{\quad A \quad} & C^{\otimes} \\ {\scriptstyle i}\big\downarrow & \nearrow{\scriptstyle \mathrm{Lan}_i A =: \int A} & \\ \mathbf{Mfld}^{\amalg}_{n,or} & & \end{array}$$

0.3 Model categories

46. Factorisation homology plays a central role in an area of maths that looks at functors called *topological quantum field theories (TQFTs)*, in that they are similarly define and themselves correspond to a subset of TQFTs.

TQFTs have been a driving force in the development of higher category theory, in particular that of (∞, n)-categories. This is largely due to a very important hypothesis called the *cobordism hypothesis*, which is a statement about how a TQFT, which gives us invariants of manifolds, is fully determined by its value on the point. If you're dealing with 1 manifolds then $\inf, 1$-category theory suffices, but if you want to look at n-manifolds then you need to be able to consider k-manifolds as morphisms between $k-1$-manifolds for all $k \leq n$. Let us introduce one particular model for (∞, n)-categories attributed to Segal.

All definitions and results of this section may be found in [4, Hovey].

Construction 47. Let category $\mathbf{Ar}(\mathcal{C})$ of morphisms of a category $\mathcal{C}$ have objects as morphisms f and g in $\mathcal{C}$ and morphisms as commuting squares.

$$\begin{array}{ccc} A & \longrightarrow & C \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle g} \\ B & \longrightarrow & D \end{array}$$

Definition 48. A *retract* of morphism f in category $\mathcal{C}$ is an morphism g such that in $\mathbf{Ar}(\mathcal{C})$ there are maps $f \to g$ and $g \to f$ that compose to the identity on f, as in the following diagram.

$$\begin{array}{ccccc} & & \overset{id_A}{\curvearrowright} & & \\ A & \longrightarrow & C & \longrightarrow & A \\ \downarrow f & & \downarrow g & & \downarrow f \\ B & \longrightarrow & D & \longrightarrow & B \\ & & \underset{id_B}{\curvearrowright} & & \end{array}$$

Notation 49. Let A and B be subcategories of $\mathcal{C}$. If all morphisms $f \in A$ have the left lifting property with respect to all morphisms in B, we shall denote this by $A_{\text{L}}\bot_{\text{R}}B$ and say that A and B are orthogonal.

Definition 50. A *model structure* on category $\mathcal{C}$ is the following data:

- three subcategories of $\mathcal{C}$: weak equivalences $\mathbf{W}$, fibrations $\mathbf{Fib}$, and cofibrations $\mathbf{Cof}$;
- two functorial factorisation systems (α, β) and (γ, δ).

Such that the following conditions hold:

(M1) (**2-out-of-3**) if any two of g, f and gf are weak equivalences then so is the third;

(M2) the categories $\mathbf{W}$, $\mathbf{Fib}$ and $\mathbf{Cof}$ are closed under retracts;

(M3) for *trivial fibrations* $\mathbf{Fib}^{\mathbf{W}} := \mathbf{Fib} \cap \mathbf{W}$ and *trivial cofibrations* $\mathbf{Cof}^{\mathbf{W}} := \mathbf{Cof} \cap \mathbf{W}$ we have that $\mathbf{Cof}_{\text{L}}\bot_{\text{R}}\, \mathbf{Fib}^{\mathbf{W}}$ and $\mathbf{Cof}^{\mathbf{W}}{}_{\text{L}}\bot_{\text{R}}\mathbf{Fib}$;

(M4) for all f, $(\alpha(f), \beta(f)) \in \mathbf{Cof} \times \mathbf{Fib}^{\mathbf{W}}$ and $(\gamma(f), \delta(f)) \in \mathbf{Cof}^{\mathbf{W}} \times \mathbf{Fib}$.

Definition 51. A *model category* $\mathcal{M}$ is a bicomplete category endowed with a model structure. An object $A \in \mathcal{M}$ is called *fibrant* if the unique map to the terminal object $A \to *$ is a fibration. Dually, an object A is called *cofibrant* if the unique map from the intial object $\circ \to A$ is a cofibration.

Construction 52. The category of topological spaces $\mathbf{Top}$ has the following model structure:

- the weak equivalences are weak homotopy equivalences;

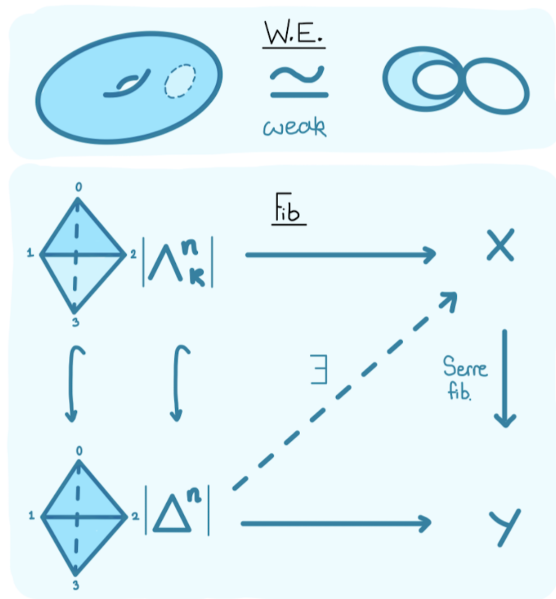


Figure 7: A model structure on Top

- the fibrations are *Serre fibrations*: maps of topological spaces that have the right lifting property with respect to the geometric realisation of all $\wedge_k^n$ horn inclusions for $n \geq 2$;
- cofibrations are all morphisms that have the left lifting property with respect to trivial fibrations.

Another example allows us to combinatorially encode homotopical information of spaces.

Construction 53. The category of simplicial sets **sSet** admits a model structure due to Quillen, denoted $\mathbf{sSet}^Q$, specified by the following data:

- weak equivalences morphisms f such that $|f|$ is a homotopy equivalence;
- fibrations are Kan fibrations;
- cofibrations are monomorphisms $m : X \to Y$ in **sSet**, families of injective maps $(X([n]) \to Y([n]))_{n \in \mathbb{N}}$.

Definition 54. The homotopy category $h_1(\mathcal{C})$ of model category $\mathcal{C}$ has objects of $\mathcal{C}$ and morphisms as homotopy classes of morphisms of $\mathcal{C}$.

0.4 Complete Segal space model of (∞,n)-categories

55. In the quasicategory model for ∞-categories all k-morphisms for k greater than 1 are invertible, so the next question is do we have a model for categories with non-invertible higher morphisms?

The definitions and results of this section may be found either in [6, Scheimbauer] or [1, Scheimbauer-Calaque].

Definition 56. An n-fold Segal space is an n-fold simplicial space

$$X : (\Delta^{op})^n \to sSet^Q$$

$X \in \mathbf{sSpaces} := Fun((\Delta^{op})^n, sSet^Q) := Fun((\Delta^{op})^n, \mathbf{Spaces})$ (which is also an $n+1$-fold simplicial set), such that:

- X is level-wise fibrant: for all $k_1, ..., k_n \in \mathbb{N}$, $X_{k_1,k_2,...,k_n}$ is a fibrant object of $\mathbf{sSet}^{\mathrm{Q}}$, i.e. $X_{k_1,k_2,...,k_n}$ is a Kan complex;
- X satisfies the Segal condition: for n=1 this is $X_n \times_{X_0} X_m \simeq X_{n+m}$. In general we have that
$$X_{i_1,...,i_{m-1},\bullet,i_{m+1},...,i_n} : \Delta^{op} \to \mathbf{sSpaces}$$
satisfies the Segal condition.

Definition 57. A Segal space satisfies *essential constancy* if and only if

$$X_{k_1,...,k_{i1},0,0,...,0} \to X_{k_1,...,k_{i1},0,k_{i+1},...,k_n}$$

is an equivalence of spaces. An n-uple Segal space with this condition is termed *n-fold.*

Definition 58. An n-uple Segal space X is *complete* if

$$X_{0,...,0} \simeq X^{\mathrm{inv}}_{1,0,...,0}$$

such that the kan complex X_0 is the 'maximal' ∞-groupoid in the class of Segal spaces that give rise to an equivalent (∞, n)-category, which means that all the invertible 1-morphisms in $X_{1,0,...,0}$ up to a choice of path are the identity.

We may recover the n-fold CSSs from a model structure on **sSpaces**.

Construction 59. • Let $\mathrm{sSet}^{\mathrm{Quil}}$ be sSet with the Quillen model structure.

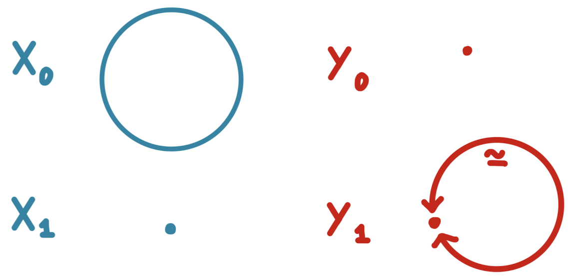


Figure 8: Two Segal spaces X (complete) and Y (not complete) that give rise to equivalent (∞, n)-categories.

- As $\text{sSet}^{\text{Quil}} \simeq \textbf{Top} := \textbf{Spaces}$, we define simplicial spaces as simplicial objects in **Spaces**.
- The model structure on $\text{sSet}^{\text{Quil}}$ extends to $\text{sSpaces} = \text{Fun}((\Delta^{op})^n, \text{sSet}^{\text{Quil}})$ where morphisms $\eta : X_\bullet \to Y_\bullet$ in sSpaces are fibrations if for every object $\phi := ([k_1], [k_2], ..., [k_n]) \in (\Delta^{op})^n$ $\eta_\phi : X_\phi \to Y_\phi$ is a fibration. This is analogous for weak equivalences.
- From fibrations and weak equivalences we may determine cofibrations **Cof** in the model structure on **sSpaces**. (This is an example of the projective model structure on sheaves.)
- Now consider the model structure on sSpaces determined by cofibrations **Cof** and Dwyer-Kan equivalences as weak equivalences.
- The fibrant objects of this model structure are n-fold complete segal spaces.

60. We will now give some exposition on interpreting an n-fold CSS as an (∞, n)-category. For an n-fold complete Segal space X we may consider the following:

Data:

- Objects are $X_{0,...,0}$
- 1-morphisms are $X_{1,0,...,0}$, 2-morphisms are $X_{1,1,0,...,0}$, etc.
- By essential constancy $X_{k,l,m,0,...,0}$ is the set of all diagrams in X which consist of the following:
  - $k + 1$ objects

- $k(l+1)$ many (primitive) 1-cells (($l+1$) k-tuples of composable 1-morphisms),
- $l(m+1)$ many primitive 2-cells,
- $k \times l \times m$ many 3-cells,
- (no higher (non-trivial) morphisms)

Composition:

- By the Segal condition

$$X_{1,1,0,\dots,0} \times_{X_{1,0,0,\dots,0}} X_{1,1,0,\dots,0} \simeq X_{1,2,0,\dots,0}$$

  which tells us that $X_{1,2}$ are the 2-tuples of vertically composable 2-morphisms, and

$$X_{1,1,0,\dots,0} \times_{X_{1,0,0,\dots,0}} X_{1,1,0,\dots,0} \simeq X_{2,1,0,\dots,0}$$

  which expresses $X_{2,1}$ as the 2-tuples of horisontally composable 2-morphisms.

- There are maps $X_{1,2,0,\dots,0} \to X_{1,1,0,\dots,0}$: horisontal composition of 2-morphisms, and $X_{2,1,0,\dots,0} \to X_{1,1,0,\dots,0}$: vertical composition of 2-morphisms and analogous generalisations to k-morphisms.

- a morphism is invertible if its image under the map which sends a pullback to π_0 of the homotopy pullback is invertible in the homotopy category of a CSS.

The below diagram shows an ordinary pullback of spaces (left) next to a homotopy pullback of spaces for Segal space X. Here the map s is the source map and $\mathrm{Hom}(x, X_0) := \{f | f \in \mathrm{Paths}(x, y), y \in X_0\}$.

$$\begin{array}{ccc} \mathrm{Hom}(x, X_0) & \longrightarrow & X_1 \\ \downarrow & \ulcorner & \downarrow s \\ \{x\} & \hookrightarrow & X_0 \end{array} \qquad \begin{array}{ccc} \pi_0(\mathrm{Hom}(x, X_0)) & \longrightarrow & X_1 \\ \downarrow & \ulcorner \ \simeq & \downarrow s \\ \{x\} & \hookrightarrow & X_0 \end{array}$$

Redundancy: The space $X_{1,1,\dots,1,0,\dots,0}$ of k-morphisms has the data of:

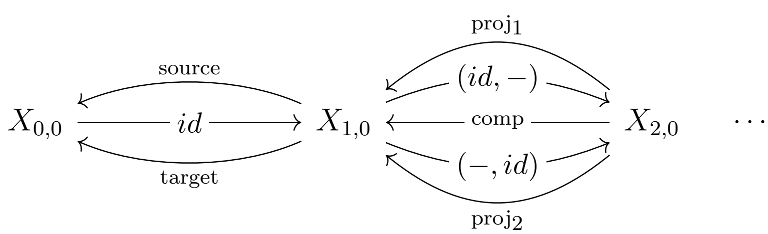
- all k-morphisms (points),
- invertible $k+1$-morphisms (paths),

- invertible $k+2$-morphisms between invertible $k+1$-morphisms (paths of paths), etc. And so X encodes the data of invertible morphisms of the corresponding (∞, n)-category many times over.

Example 61. We give the Morita bicategory as an n-fold Segal space.

- $X_{0,0}$: the objects are rings
- $X_{1,0}$: morphisms are bimodules;
- $X_{0,1}$: vertical 2-morphisms are rings as bimodules over themselves (which gives us the essential constancy condition);
- $X_{1,1}$: horisontal (usual) 2-morphisms are bimodule homomorphisms.

In the following diagram we specify the data of the morphisms in $X_\bullet, 0$.



Segal space $X_{1,\bullet}$ is analogous. Segal space $X_{2,\bullet}$ also has the same diagram but the source and target maps give 2-tuples of 1-morphisms, and so on.

Conclusion

In this exposition we have laid a lot of the necessary groundwork to begin to tackle Schiembauer and Calaque's paper on the (∞, n)-category of bordism. This construction was made to bridge the gaps in an important section Lurie's sketch of a proof for the cobordism hypothesis.

As the reader has begun to see, the combinatorial formulation of homtopy is very powerful to aid the organisation of infinite amounts of homotopical coherence data. This in turn allows us to develop ever more subtle and powerful invariants of spaces and specifically manifolds, which are incredibly useful to physics and other areas of mathematics.

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